University of Stuttgart
Institute for Structural Mechanics

An objective and path-independent geometrically non-linear Reissner-Mindlin shell formulation

WCCM-APCOM
Yokohama

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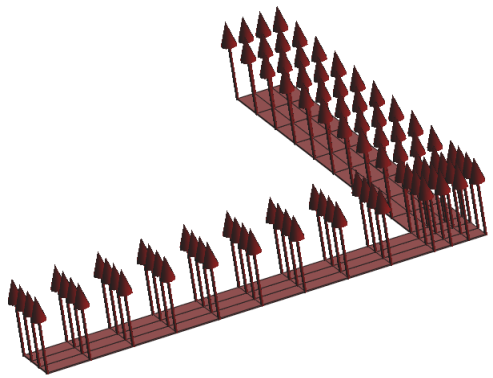
Alexander Müller, Manfred Bischoff

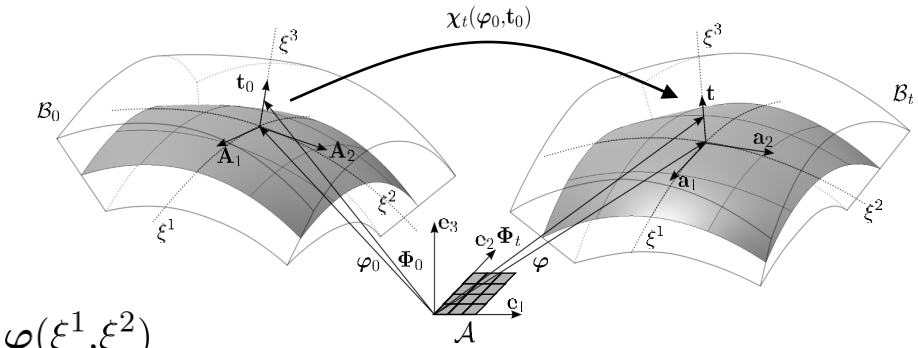


Baustatik und Baudynamik

Kinematics of a nonlinear Reissner-Mindlin shell model

Baustatik und Baudynamik

Kinematic assumptions



- Geometry approximation:

$$\mathbf{x} = \mathbf{\Phi}_t(\xi^1,\xi^2,\xi^3) = \boldsymbol{\varphi}(\xi^1,\xi^2) + \xi^3\mathbf{t}(\xi^1,\xi^2)$$

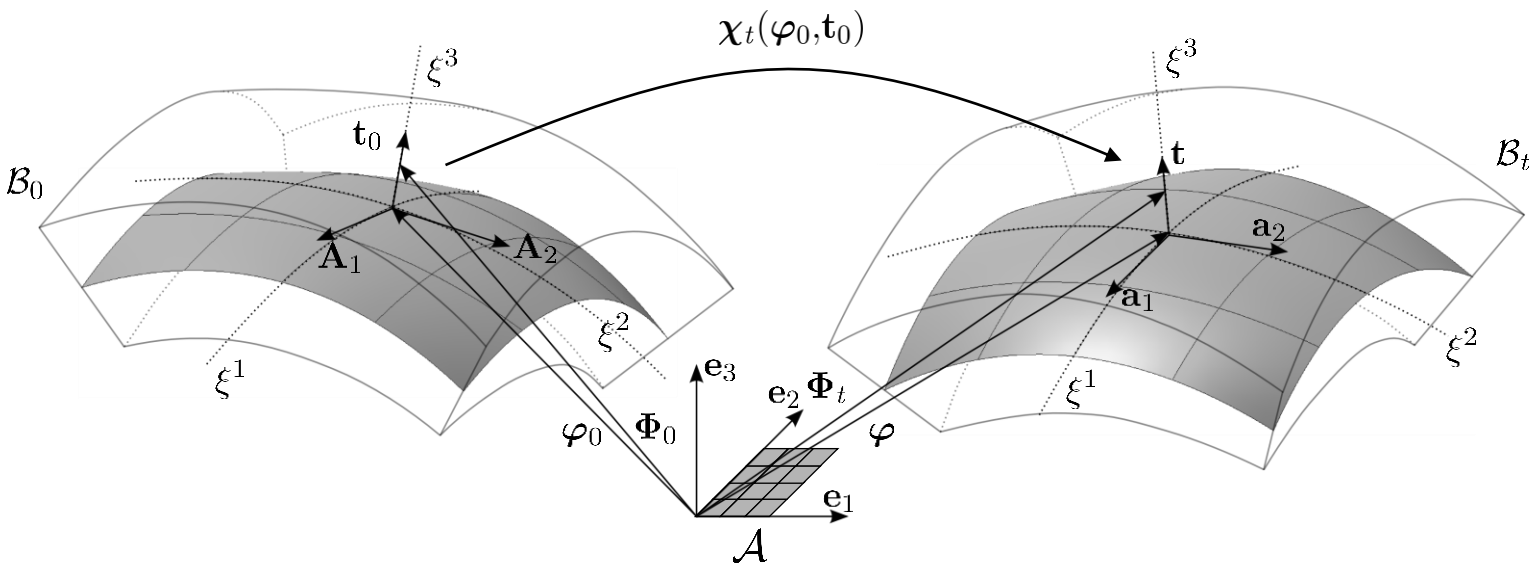
- First order transverse shear effects are taken into account:
  - The director field $\mathbf{t}(\xi^1,\xi^2)$ is independent of the midsurface field $\boldsymbol{\varphi}(\xi^1,\xi^2)$ (in contrast to Kirchhoff-Love)
- Thickness change is not contained in kinematic description:
  - The director is a unit vector, i.e. $\mathbf{t} : \mathcal{A} \to \mathcal{S}^2 \subset \mathbb{R}^3$ $\qquad \mathcal{S}^2 = \{\mathbf{x} \in \mathbb{R}^3 \,||\, \mathbf{x}\cdot\mathbf{x} = 1\}$

Discretization

- Discretization of the midsurface field $\boldsymbol{\varphi} : \mathcal{A} \to \mathbb{R}^3$ trivial since it maps onto a vector space $\mathbb{R}^3$
- Discretization and parametrization of the director field $\mathbf{t} : \mathcal{A} \to \mathcal{S}^2$ difficult due to the nonlinear space $\mathcal{S}^2$

Director fields

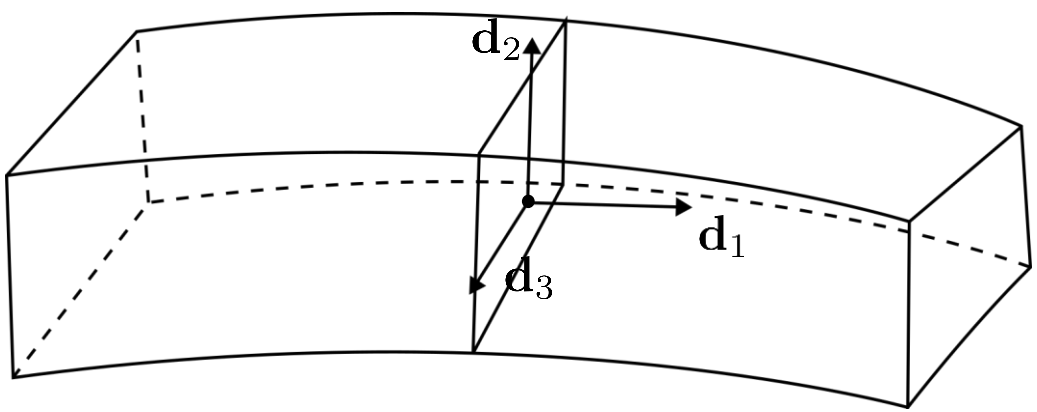
Example 1: Geometrically nonlinear Reissner-Mindlin shell



$\mathbf{t}(\xi^1, \xi^2)$ $\qquad$ $\mathbf{t} : \mathcal{A} \to \mathcal{S}^2 \subset \mathbb{R}^3$

Director fields

Example 2: Geometrically nonlinear beam

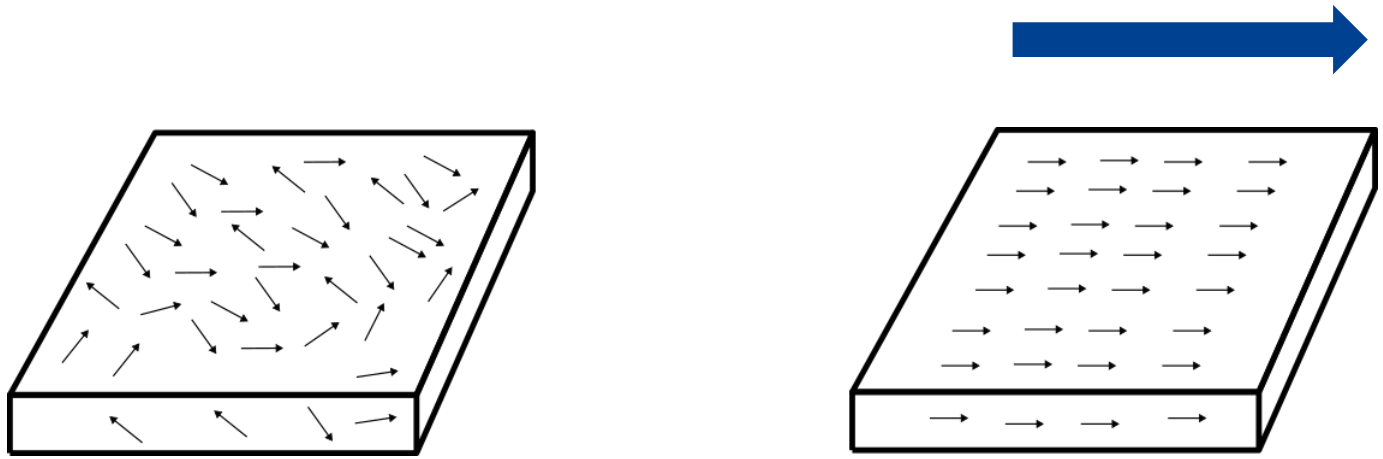


$$\mathbf{R} = [\mathbf{d}_1 \ \mathbf{d}_2 \ \mathbf{d}_3] \quad \mathbf{d}_i \cdot \mathbf{d}_j = \delta_{ij}$$

$$\mathbf{R} \in \mathcal{SO}(3)$$

Director fields

Example 3: Magnetic Maxwell equations in matter



$$\mathbf{m}(\xi^1,\xi^2,\xi^3)$$

$$\mathbf{m} : \mathcal{A} \to \mathcal{S}^2 \subset \mathbb{R}^3$$

Outline

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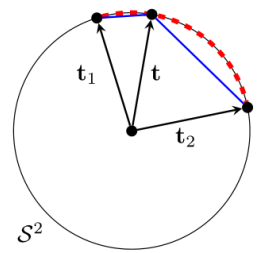
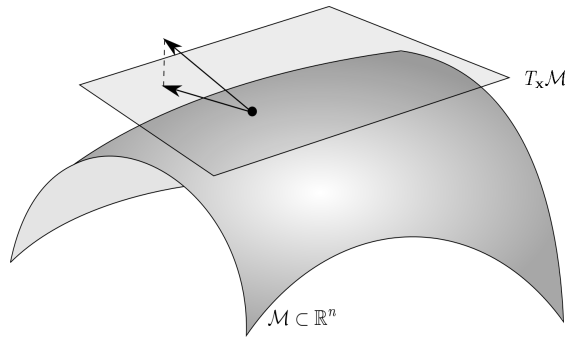
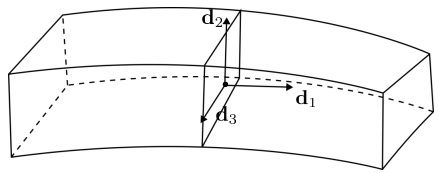
Interpolation on manifolds (FE for manifolds)



Optimization on manifolds



Open questions/Outlook



Interpolation: Review of historic approaches

(for Reissner-Mindlin shell)

Interpolation: Review of existing approaches

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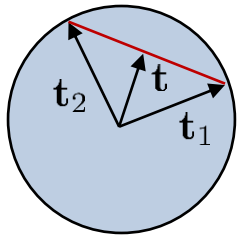
Example 1: Angles

- The unit sphere can be parameterized with an angle pair $(\alpha,\beta) \rightarrow \quad \mathbf{t} = \mathbf{R}(\alpha,\beta)\mathbf{t}_0, \;\; \mathbf{R}(\alpha,\beta) \in \mathcal{SO}(3)$

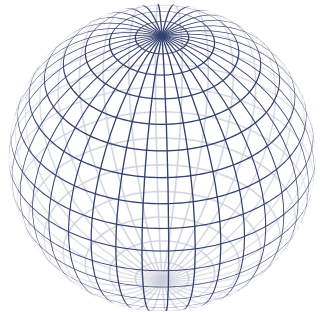
RAMM (1976), ARGYRIS(1982), BAŞAR ET AL(1992), WRIGGERS & GRUTTMANN (1993)

  - The resulting interpolation is, e.g. RAMM (1976)

$$\mathbf{t} = \sum_{A=1}^{n} N^A(\xi)\mathbf{R}_A(\alpha^A,\beta^A)\mathbf{t}_0^A, \quad \mathbf{t}_A = \mathbf{R}_A\mathbf{t}_0^A$$



  - Singularities, violates objectivity, unit length constraint violated



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Interpolation: Review of existing approaches

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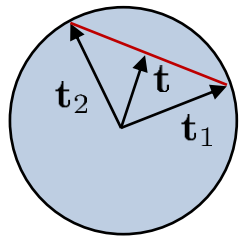
Example 2: Direct interpolation of the nodal directors

- Standard interpolation formula for finite elements in vector spaces
  - Simple straightforward interpolation

HUGHES & LIU (1981), BATHE & BOLOURCHI (1980), BETSCH & STEINMANN (2002), BENSON ET AL (2010)

$$\mathbf{t} = \sum_{A=1}^{n} N^A(\xi)\mathbf{t}_A$$

  - Interpolated value does not lie on the unit sphere
  - Objective



$\mathbf{t} \notin \mathcal{S}^2$

Interpolation: Review of existing approaches

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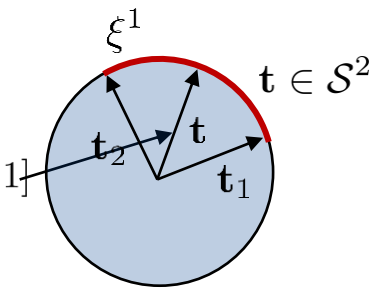
Example 3: Generalized Spherical Linear interpolation (SLERP)

- SLERP: Interpolation between two unit vectors $\mathbf{t}_1, \mathbf{t}_2$

$$\mathbf{t} = \mathrm{SLERP}(\mathbf{t}_1, \mathbf{t}_2, \xi^1) = \frac{\sin((1-\xi^1)\arccos(\mathbf{t}_1 \cdot \mathbf{t}_2))}{\sin(\arccos(\mathbf{t}_1 \cdot \mathbf{t}_2))}\mathbf{t}_1 + \frac{\sin(\xi^1 \arccos(\mathbf{t}_1 \cdot \mathbf{t}_2))}{\sin(\arccos(\mathbf{t}_1 \cdot \mathbf{t}_2))}\mathbf{t}_2, \quad \xi^1 = [0,1]$$



Directors have unit length

- Generalization for four vectors in 2D

$$\mathbf{t} = \mathrm{SLERP}[\mathrm{SLERP}(\mathbf{t}_1, \mathbf{t}_2, \xi^1), \mathrm{SLERP}(\mathbf{t}_3, \mathbf{t}_4, \xi^1), \xi^2]$$ AREIAS (2013)

  - Complicated interpolation
  - Objective

Interpolation: Review of existing approaches

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Example 4: Interpolating increments

- Only incremental rotations are interpolated SIMO ET AL (1990), BÜCHTER & RAMM (1992), DORNISCH ET AL (2016)

$$\Delta\boldsymbol{\theta} = \sum_i N^i(\xi,\eta)\Delta\boldsymbol{\theta}_i$$

- Rotation matrices are updated at each IP SIMO ET AL (1990), DORNISCH ET AL (2016)

$$\mathbf{R}_{IP}^{k+1} = \exp(\Delta\hat{\boldsymbol{\theta}})\mathbf{R}_{IP}^{k}$$

→ Non-objective and path-dependent for elastic problems as proven in CRISFIELD & JELENIĆ (1999)

- Directors at GPs have unit length

Interpolation: Desired Properties

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- **Objective**
- **Singularity-free**
- **Interpolated value stays on the manifold**
- **Useable for arbitrary polynomial order**
- **Generalizable from 1D**
- **Invariant to node numbering**
- **No artificial path-dependence**

Suitable interpolation schemes for the director field

Suitable interpolation schemes for the director field

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Geodesic Finite Elements SANDER (2012)

GFE define a class of finite elements to interpolate on manifold

- Consider the following interpolation scheme

$$\mathbf{x}^*(\boldsymbol{\xi}, \mathbf{x}_i) = \arg\min_{\mathbf{x}\in\boxed{\mathbb{R}^n}} f(\mathbf{x}, \boldsymbol{\xi}; \mathbf{x}_i) \qquad f(\mathbf{x}, \boldsymbol{\xi}; \mathbf{x}_i) = \sum_{i=1}^{n} N^i(\boldsymbol{\xi})||\mathbf{x}_i - \mathbf{x}||^2$$

- Since: $\dfrac{\partial f(\mathbf{x}, \boldsymbol{\xi}; \mathbf{x}_i)}{\partial \mathbf{x}} \overset{!}{=} \mathbf{0} = \sum_{i=1}^{n} N^i(\boldsymbol{\xi})(\mathbf{x}_i - \mathbf{x}) = \sum_{i=1}^{n} N^i(\boldsymbol{\xi})\mathbf{x}_i - \underbrace{\sum_{i=1}^{n} N^i}_{=1}\mathbf{x}$ → $\mathbf{x}^* = \sum_{i=1}^{n} N^i(\boldsymbol{\xi})\mathbf{x}_i$

Identical to standard interpolation!

- Euclidean distance can be generalized for the manifold $\mathcal{M}$

$$\mathbf{x}^* = \arg\min_{\mathbf{x}\in\boxed{\mathcal{M}}} \sum_{i=1}^{n} N^i(\boldsymbol{\xi})\,\mathrm{dist}(\mathbf{x}_i, \mathbf{x})^2_{\mathcal{M}} = \mathbf{x}_{GP}$$

- objective since distances are a priori rotational invariant
- Directors have unit length
- Implicit interpolation by minimization problem → Nonlinear minimization problem at each integration point

Suitable interpolation schemes for the director field

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Projection-Based Finite Elements GROHS ET AL (2019)

Projection-based interpolation is a special kind of geodesic finite elements

- In contrast to the GFE definition

$$\mathbf{x}^* = \arg\min_{\mathbf{x}\in\mathcal{M}} \sum_{i=1}^{n} N^i(\boldsymbol{\xi})\,\mathrm{dist}(\mathbf{x}_i, \mathbf{x})^2_{\boxed{\mathcal{M}}} = \mathbf{x}_{GP}$$

- PB finite elements use the distance of the embedding space

$$\mathbf{x}^* = \arg\min_{\mathbf{x}\in\mathcal{M}} \sum_{i=1}^{n} N^i(\boldsymbol{\xi})\,\mathrm{dist}(\mathbf{x}_i, \mathbf{x})^2_{\boxed{\mathbb{R}^n}} = \mathbf{x}_{GP}$$

- objective since distances are a priori rotational invariant
- Directors have unit length
- Implicit interpolation by minimization problem

Suitable interpolation schemes for the director field

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What does all that mean for the unit sphere?

Example with two directors:

$$N^1(\xi) = 1 - \xi, \quad N^2(\xi) = \xi$$

- **NFE**/Standard interpolation:

$$\mathbf{t}_{\mathrm{GP}} = \sum_{i=1}^{n} N^i(\xi)\mathbf{t}_i$$

- Projection-Based (**PBFE**): GROHS ET AL (2019)

$\mathbf{t}_1$ $\mathbf{t}_2$

$$\mathbf{t}_{\mathrm{GP}} = \arg\min_{\mathbf{t}\in\mathcal{S}^2} \sum_{i=1}^{n} N^i(\xi)\,\mathrm{dist}(\mathbf{t}_i, \mathbf{t})^2_{\mathbb{R}^n}$$

$$= \arg\min_{\mathbf{t}\in\mathcal{S}^2} \sum_{i=1}^{n} N^i(\xi)||\mathbf{t}_i - \mathbf{t}||^2 = \frac{\sum_{i=1}^{n} N^i(\xi)\mathbf{t}_i}{||\sum_{i=1}^{n} N^i(\xi)\mathbf{t}_i||}$$

- Geodesic Finite Elements (**GFE**): SANDER (2012)

$$\mathbf{t}_{\mathrm{GP}} = \arg\min_{\mathbf{t}\in\mathcal{S}^2} \sum_{i=1}^{n} N^i(\xi)\,\mathrm{dist}(\mathbf{t}_i, \mathbf{t})^2_{\mathcal{S}^2}$$

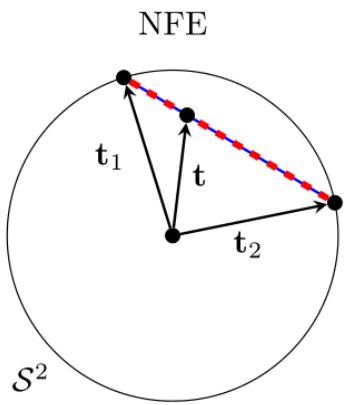
$$= \arg\min_{\mathbf{t}\in\mathcal{S}^2} \sum_{i=1}^{n} N^i(\xi)\arccos(\mathbf{t}_i \cdot \mathbf{t})^2$$

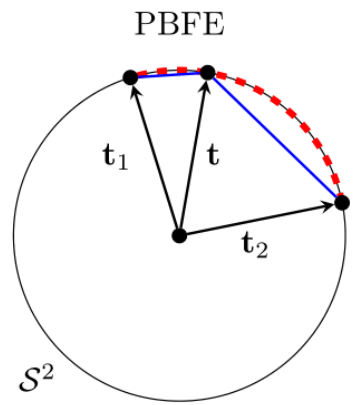
Suitable interpolation schemes for the director field

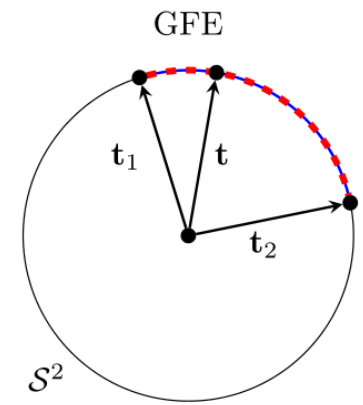
Baustatik und Baudynamik

What does all that mean for the unit sphere?

Example with two directors:







$$\mathbf{t}_{\mathrm{GP}} = \arg\min_{\mathbf{t}\in\mathbb{R}^n} \sum_{i=1}^{n} N^i(\boldsymbol{\xi})||\mathbf{t}_i - \mathbf{t}||^2 = \sum_{i=1}^{n} N^i(\boldsymbol{\xi})\mathbf{t}_i$$

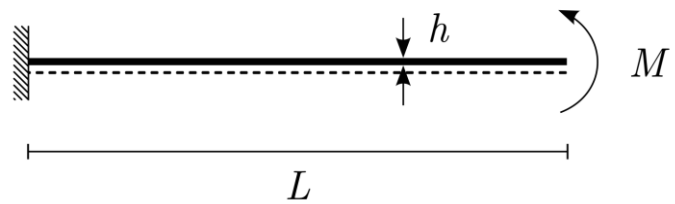
$$\mathbf{t}_{\mathrm{GP}} = \arg\min_{\mathbf{t}\in\mathcal{S}^2} \sum_{i=1}^{n} N^i(\boldsymbol{\xi})||\mathbf{t}_i - \mathbf{t}||^2 = \frac{\sum_{i=1}^{n} N^i(\boldsymbol{\xi})\mathbf{t}_i}{||\sum_{i=1}^{n} N^i(\boldsymbol{\xi})\mathbf{t}_i||}$$

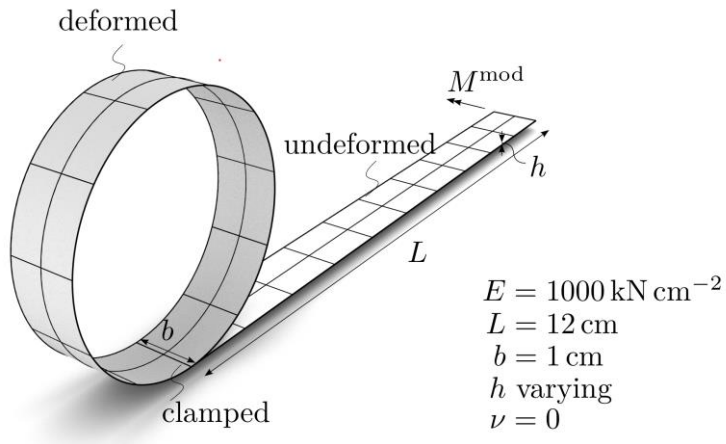
$$\mathbf{t}_{\mathrm{GP}} = \arg\min_{\mathbf{t}\in\mathcal{S}^2} \sum_{i=1}^{n} N^i(\boldsymbol{\xi}) \arccos(\mathbf{t}_i \cdot \mathbf{t})^2$$

Comparison of interpolation schemes

Interpolation: Numerical experiments

Roll-up of clamped beam





$$\Pi(\boldsymbol{\varphi}, \mathbf{t}) = \int_{\mathcal{B}} \psi_{\mathrm{strain}}(\boldsymbol{\varphi}, \mathbf{t})\,\mathrm{d}\,V + \Pi_{\mathrm{ext}}(\boldsymbol{\varphi}, \mathbf{t})$$

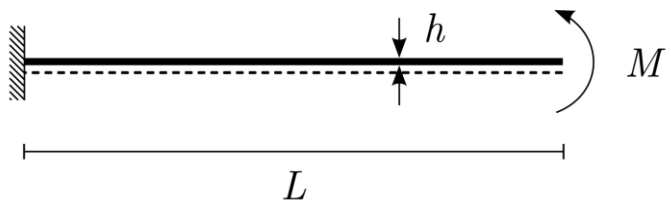
$$\{\boldsymbol{\varphi}^*, \mathbf{t}^*\} = \arg\left\{\min_{\boldsymbol{\varphi}\in\mathbb{R}^d}\ \min_{\mathbf{t}\in\mathcal{S}^{d-1}} \Pi(\boldsymbol{\varphi}, \mathbf{t})\right\}.$$

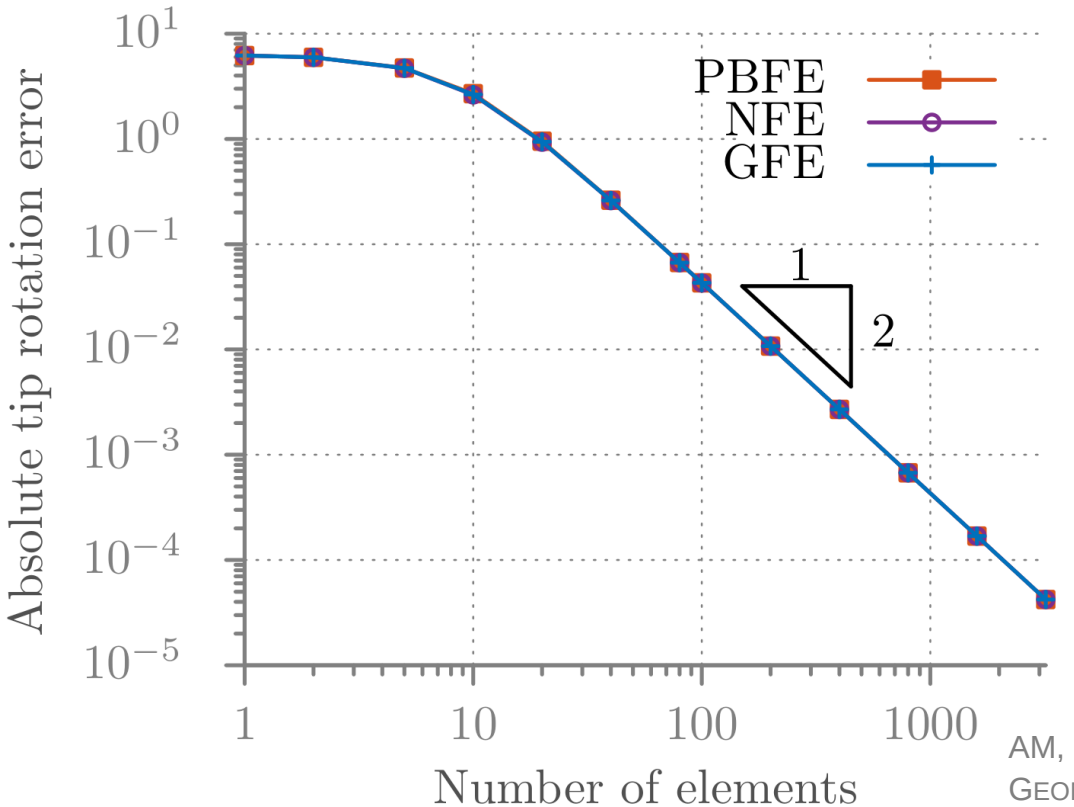
AM, Bischoff (2022): A Consistent Finite Element Formulation of the Geometrically Non-linear Reissner-Mindlin Shell Model, DOI

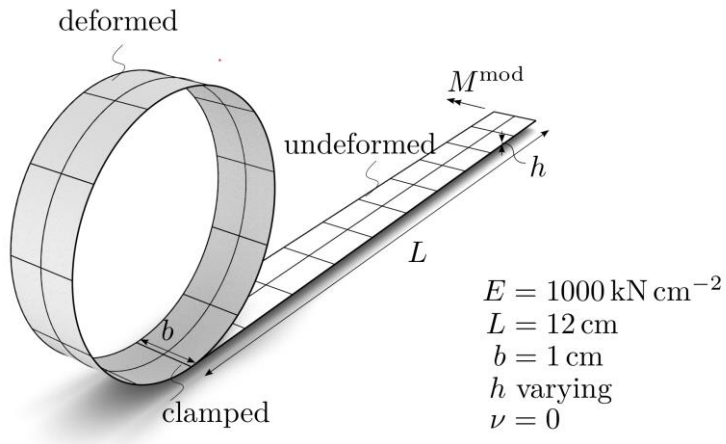
Interpolation: Numerical experiments

Roll-up of clamped beam

- Reference plane → Reference interpolation identical
- Q1 shell elements, C^0-continuity between elements







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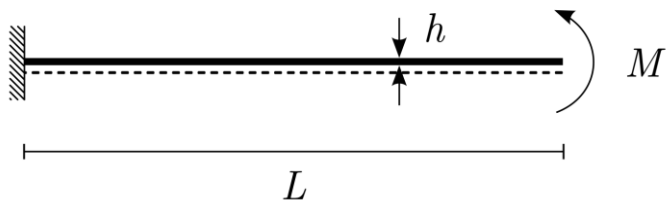
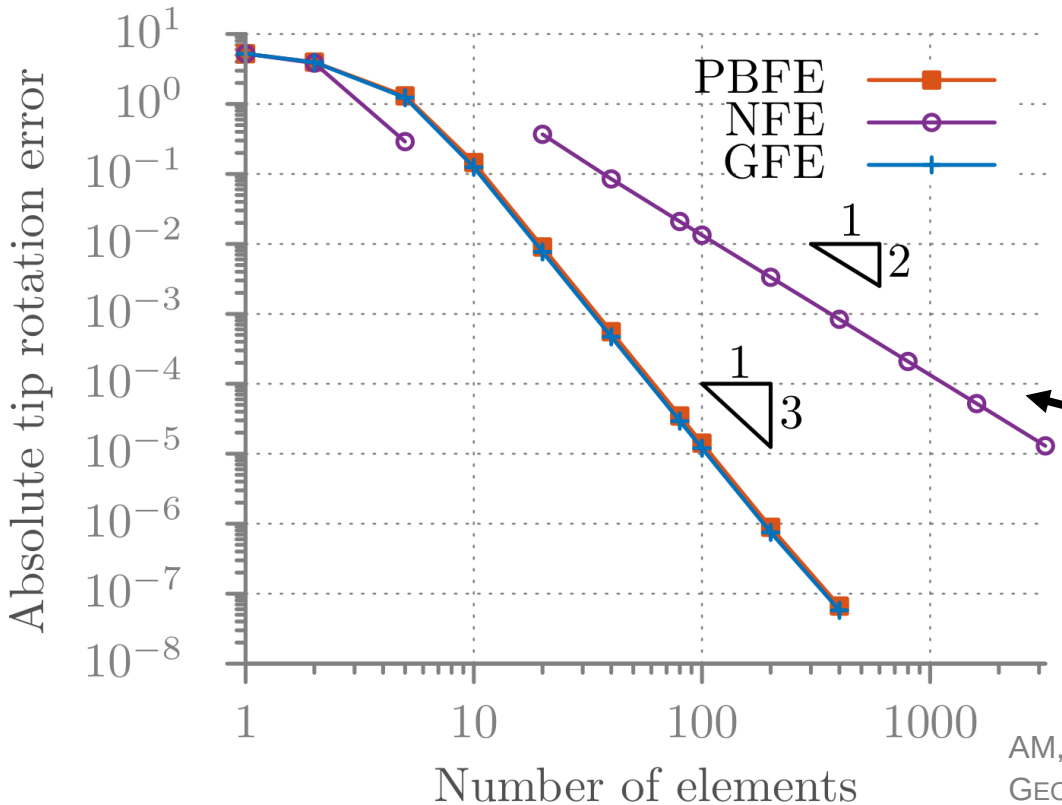
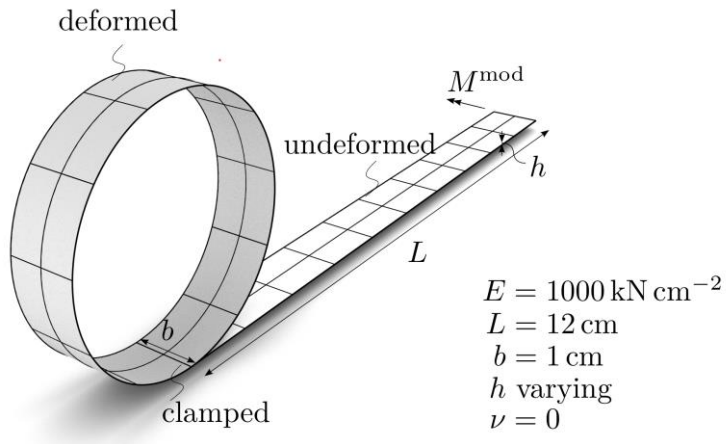
Interpolation: Numerical experiments

Roll-up of clamped beam

- Reference plane → Reference interpolation identical
- Quadratic $p = 2$ B-spline shell elements, C^1-continuity between elements







Convergence order degenerates!

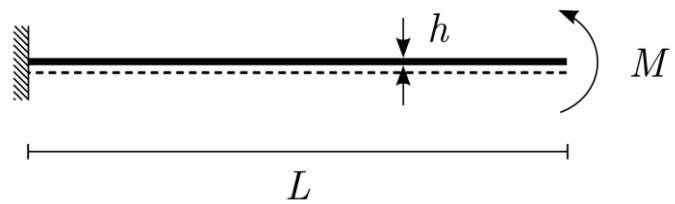
AM, Bischoff (2022): A Consistent Finite Element Formulation of the Geometrically Non-linear Reissner-Mindlin Shell Model, DOI

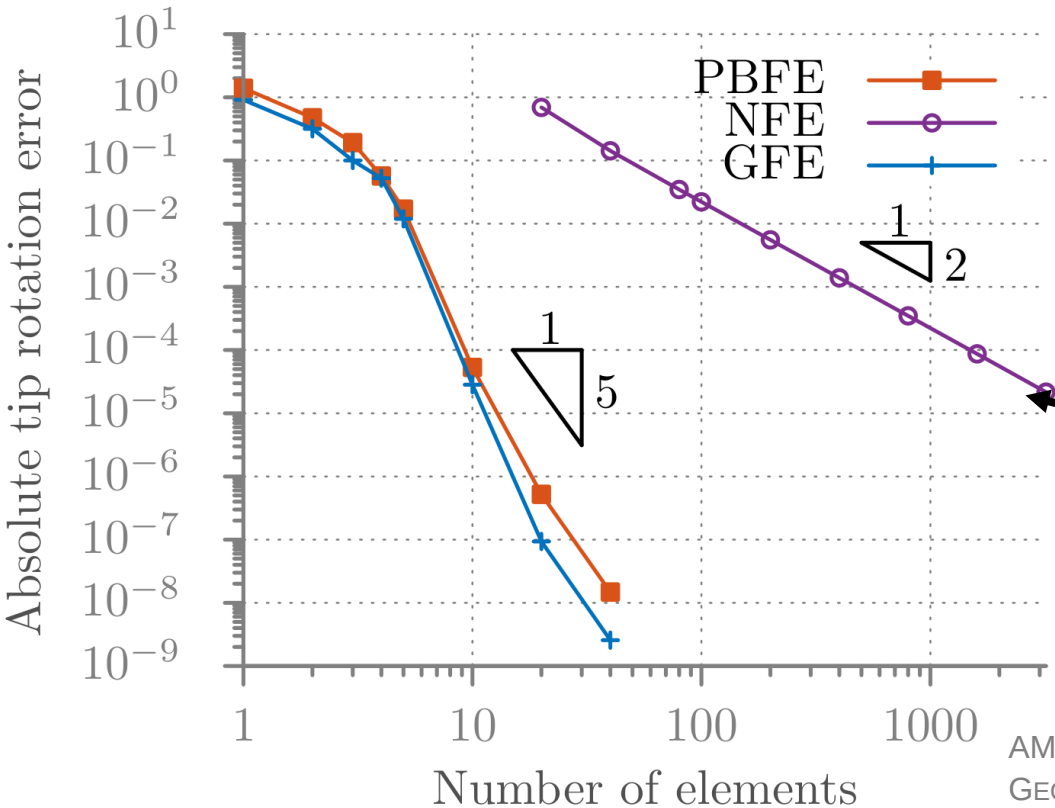
Interpolation: Numerical experiments

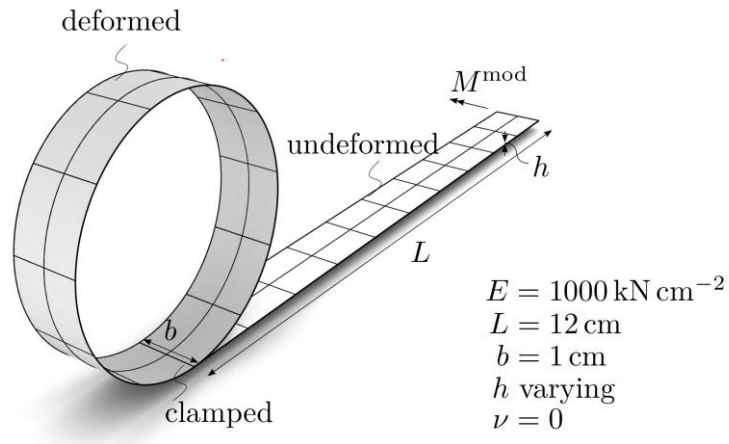
Baustatik und Baudynamik

Roll-up of clamped beam

- Reference plane → Reference Interpolation identical
- Quartic $p = 4$ B-spline shell elements, C^3-continuity between elements







Convergence order degenerates!

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Interpolation: Desired Properties

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Geometric (PBFE and GFE) satisfy all the desired properties

- Geometric (PBFE and GFE) satisfy all the desired properties
  - **Objective**
  - **Singularity-free**
  - **Interpolated value stays on the manifold**
  - **Useable for arbitrary polynomial order**
  - **Generalizable from 1D**
  - **Invariant to node numbering**
  - **No artificial path-dependence**
- Drawbacks
  - Maybe expensive to evaluate

Interlude: Function spaces

Interlude: Test function spaces

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- First order geometric finite element functions satisfy $V_h^M(\Omega) \subset H^1(\Omega, M)$

- Discrete test functions live in the tangent bundle $\delta\mathbf{u} \in T_{\mathbf{u}} V_h^M(\Omega)$

- Manifold

$$\mathbf{u} \qquad V_h^M(\Omega) \subset H^1(\Omega, M)$$

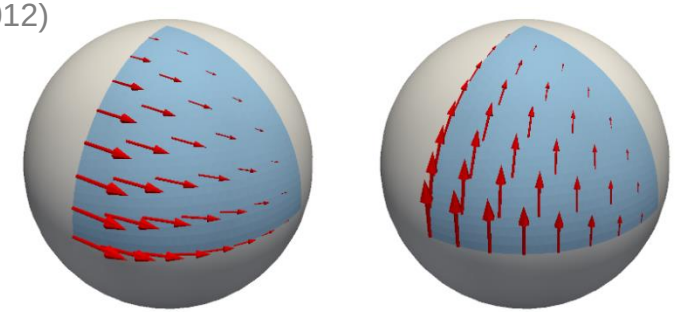
$$\delta\mathbf{u} \qquad T_{\mathbf{u}} V_h^M(\Omega)$$

- Vector space

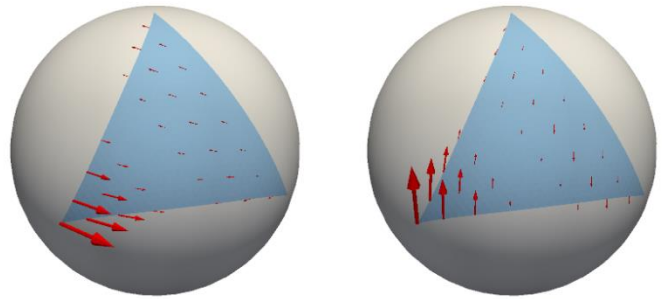
$$\mathbf{u} = \sum_{i=1}^{n} N^i(\xi)\mathbf{u}_i \qquad V_h^{\mathbb{R}^n}(\Omega) \subset H^1(\Omega, \mathbb{R}^n)$$

$$\delta\mathbf{u} = \sum_{i=1}^{n} N^i(\xi)\mathbf{u}_i \qquad T_{\mathbf{u}} V_h^{\mathbb{R}^n}(\Omega) = V_h^{\mathbb{R}^n}(\Omega) \subset H^1(\Omega, \mathbb{R}^n)$$

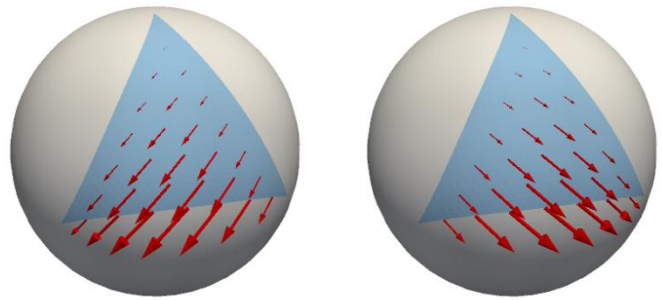
SANDER O (2012)



(A) First-order



(B) Second-order, vertex degree of freedom



(C) Second-order, edge degree of freedom

SANDER O (2016)

Outline

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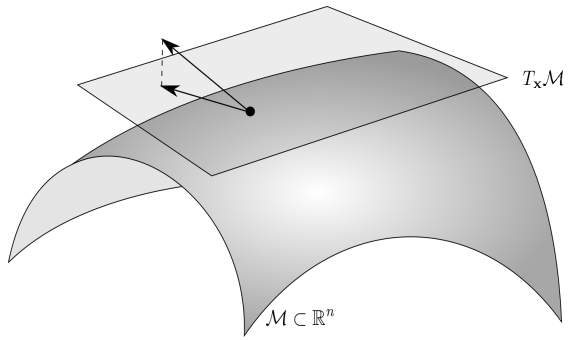
Interpolation on manifolds
(FE for manifolds)

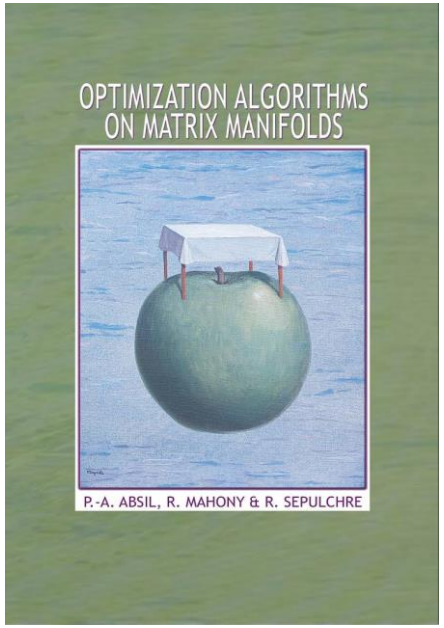
Optimization on manifolds



Open questions/Outlook

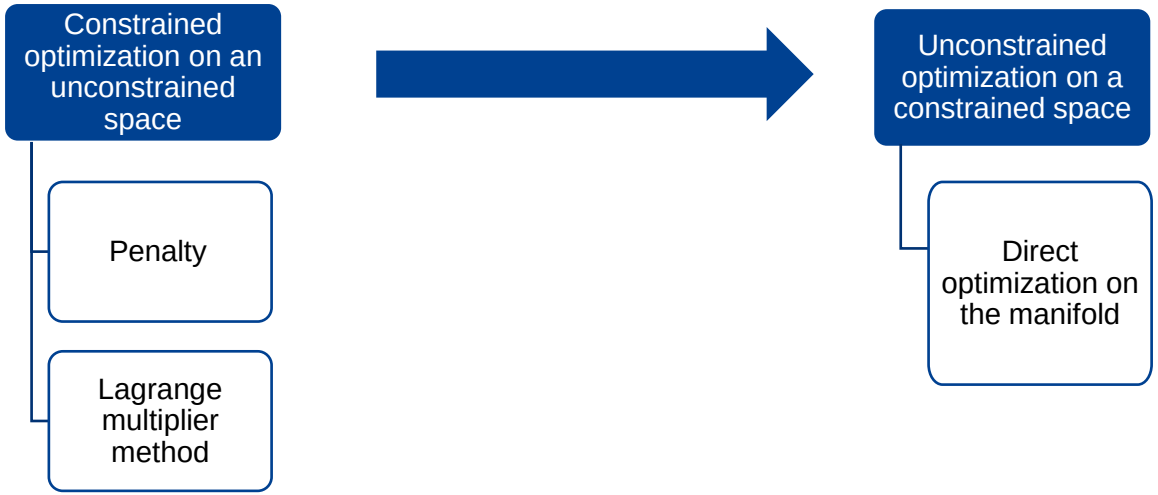
Optimization on manifolds

Literature



Absil PA, Mahony R, Sepulchre R (2008) Optimization Algorithms on Matrix Manifolds. Princeton University Press, doi:10.1515/9781400830244

Boumal N (2020) An introduction to optimization on smooth manifolds. Available online, Link

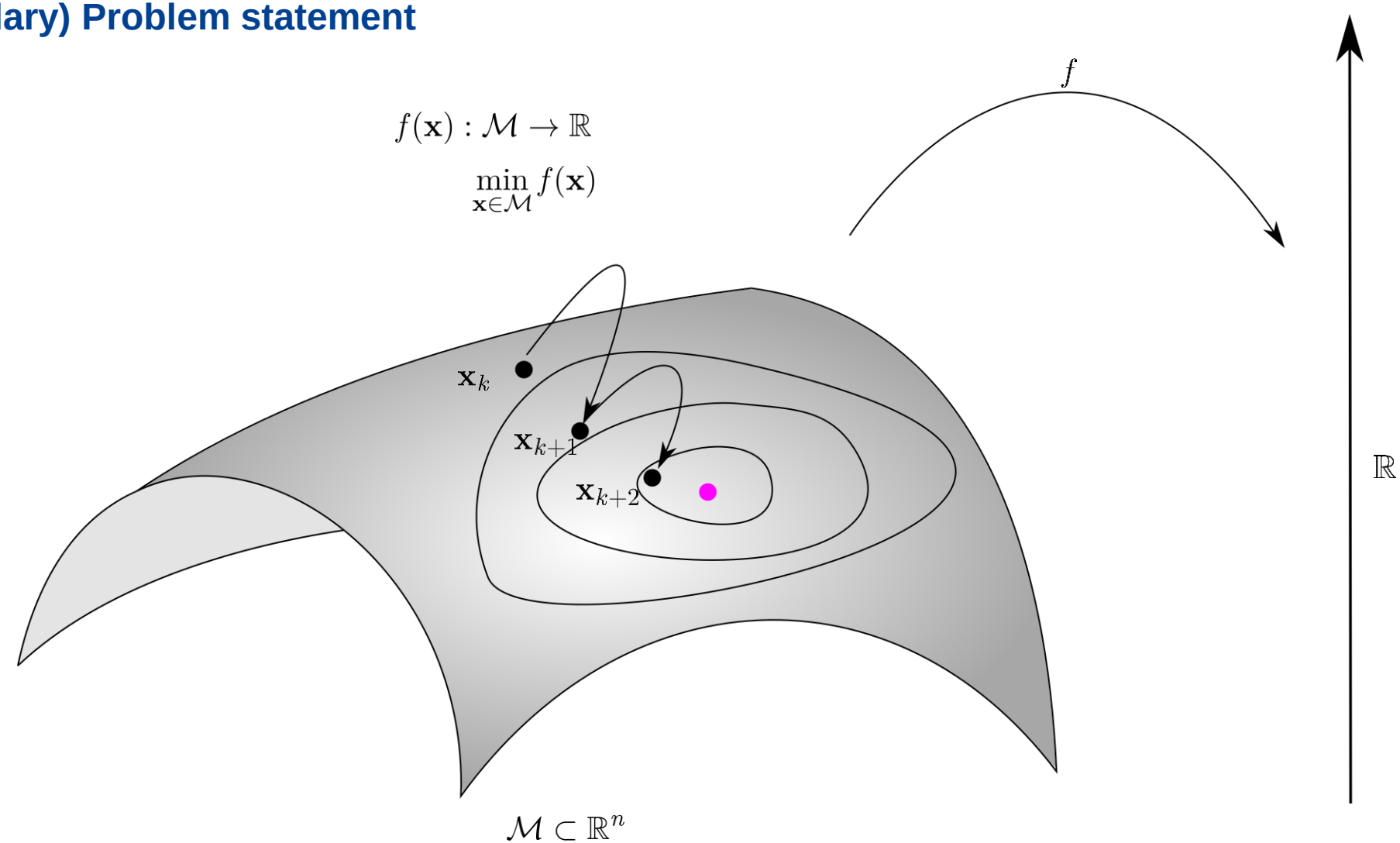


Optimization on manifolds

(Exemplary) Problem statement

$$f(\mathbf{x}) : \mathcal{M} \to \mathbb{R}$$

$$\min_{\mathbf{x} \in \mathcal{M}} f(\mathbf{x})$$



Optimization on manifolds

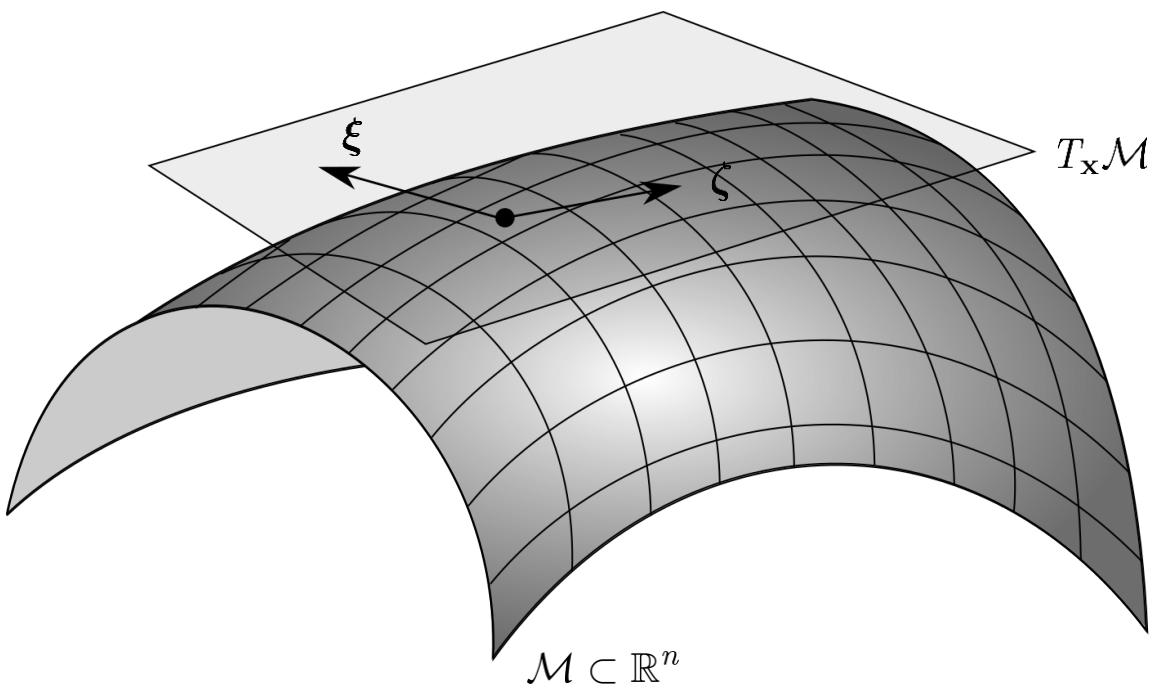
Baustatik und Baudynamik

Riemannian submanifolds

$$f(\mathbf{x}) : \mathcal{M} \to \mathbb{R} \qquad \mathcal{M} \subset \mathbb{R}^n$$

Metric of the embedding $\bar{g}_{\mathbf{x}}$ induces metric $g_{\mathbf{x}}$ on $\mathcal{M}$

$$g_{\mathbf{x}}(\boldsymbol{\xi},\boldsymbol{\zeta}) = \bar{g}_{\mathbf{x}}(\boldsymbol{\xi},\boldsymbol{\zeta}), \quad \boldsymbol{\xi},\boldsymbol{\zeta} \in T_{\mathbf{x}}\mathcal{M}$$

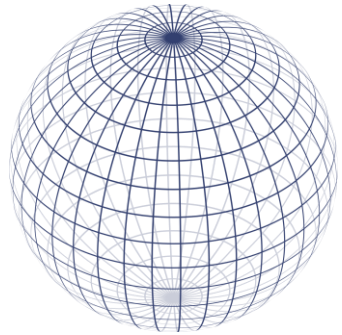


Spherical coordinates [edit]

We may define a coordinate system in an n-dimensional Euclidean space which is ar
of a radial coordinate r, and $n-1$ angular coordinates $\varphi_1, \varphi_2, \ldots \varphi_{n-1}$, where the an
$[0,360)$ degrees). If x_i are the Cartesian coordinates, then we may compute $x_1, \ldots x_n$

$$\begin{aligned}
x_1 &= r\cos(\varphi_1) \\
x_2 &= r\sin(\varphi_1)\cos(\varphi_2) \\
x_3 &= r\sin(\varphi_1)\sin(\varphi_2)\cos(\varphi_3) \\
&\vdots \\
x_{n-1} &= r\sin(\varphi_1)\cdots\sin(\varphi_{n-2})\cos(\varphi_{n-1}) \\
x_n &= r\sin(\varphi_1)\cdots\sin(\varphi_{n-2})\sin(\varphi_{n-1}).
\end{aligned}$$

WIKIPEDIA: N-SPHERE



Riemannian gradient

Optimization on manifolds

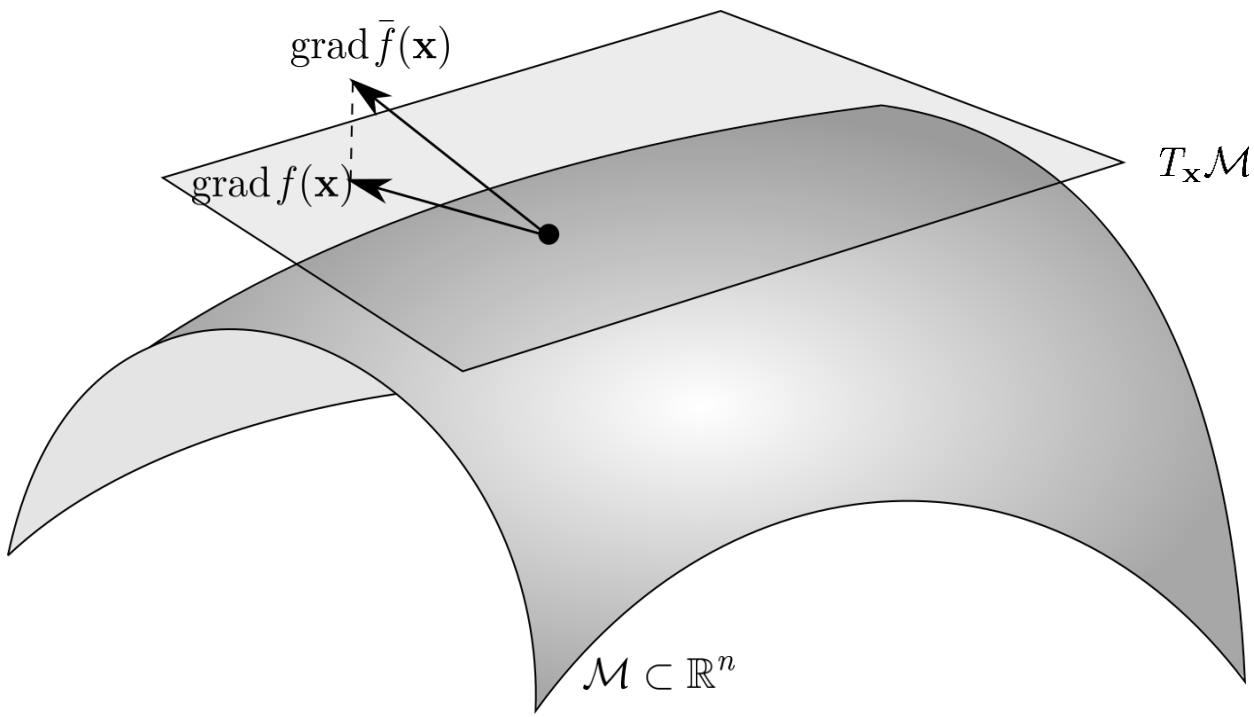
Riemannian Gradient: submanifolds

$$f(\mathbf{x}) : \mathcal{M} \to \mathbb{R} \qquad \bar{f}(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$$

"For Riemannian submanifolds, the Riemannian gradient is the orthogonal projection of the "classical" gradient to the tangent spaces."

BOUMAL N (2020) AN INTRODUCTION TO OPTIMIZATION ON SMOOTH MANIFOLDS.

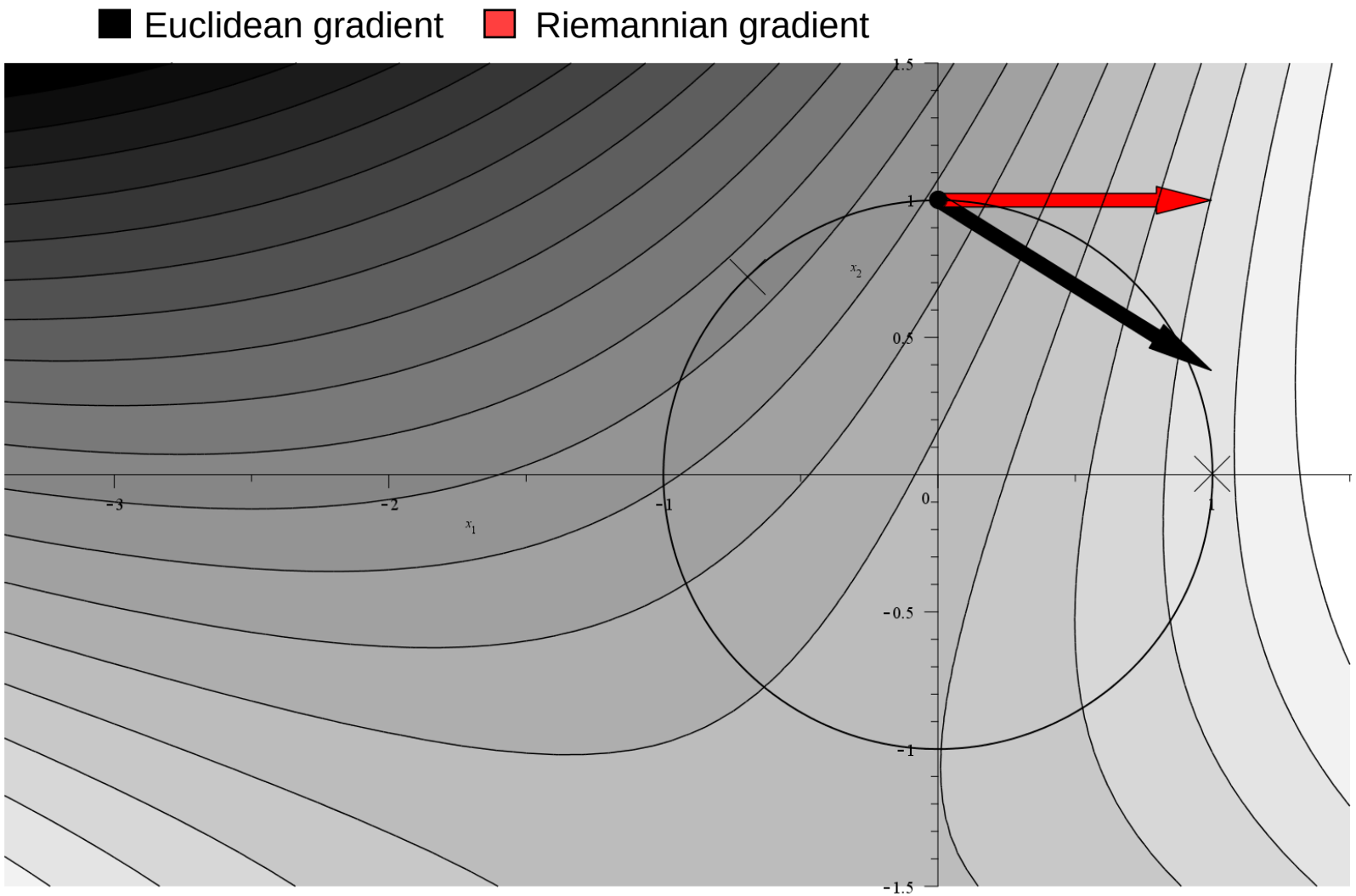
$$\operatorname{grad} f(\mathbf{x}) = P_{\mathbf{x}} \operatorname{grad} \bar{f}(\mathbf{x})$$



- No parametrization
- No artificial singularities
- Simple linearization

Optimization on manifolds

Toy problem, gradient and Riemannian gradient



Riemannian Hessian

Optimization on Manifolds

Riemannian Hessian: submanifolds

$$f(\mathbf{x}) : \mathcal{M} \to \mathbb{R} \qquad \bar{f}(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$$

Levi-Civita connection: $\nabla_{\boldsymbol{\eta}_\mathbf{x}} \boldsymbol{\xi} = P_\mathbf{x} \bar{\nabla}_{\boldsymbol{\eta}_\mathbf{x}} \boldsymbol{\xi} = P_\mathbf{x} \mathrm{D}_{\boldsymbol{\eta}_\mathbf{x}} \boldsymbol{\xi} \qquad \boldsymbol{\xi}, \boldsymbol{\eta} \in T_\mathbf{x}\mathcal{M}$

$$\mathrm{Hess}\, f(\mathbf{x})\boldsymbol{\eta} = P_\mathbf{x}\, \mathrm{Hess}\, \bar{f}(\mathbf{x}) P_\mathbf{x} \boldsymbol{\eta} + W_\mathbf{x}(\boldsymbol{\eta}, P_\mathbf{x}^\perp\, \mathrm{grad}\, \bar{f}(\mathbf{x}))$$

"[…] This shows that, for Riemannian submanifolds of Euclidean spaces, the Riemannian Hessian is the projected Euclidean Hessian plus a correction term which depends only on the normal part of the Euclidean gradient."

BOUMAL N (2020) AN INTRODUCTION TO OPTIMIZATION ON SMOOTH MANIFOLDS.

Algebraic optimization on manifolds

Reduction of dimensions

Gradient and Hessian, example unit sphere $\mathcal{S}^{n-1}$

$$P_{\mathbf{x}} = \mathbf{I} - \mathbf{x} \otimes \mathbf{x}$$

$$\operatorname{grad} \Pi(\mathbf{x}) = P_{\mathbf{x}} \operatorname{grad} \bar{\Pi}(\mathbf{x}) \qquad \operatorname{Hess} \Pi(\mathbf{x}) = P_{\mathbf{x}} \operatorname{Hess} \bar{\Pi}(\mathbf{x}) - (\mathbf{x}^T \operatorname{grad} \bar{\Pi}(\mathbf{x}))\mathbf{I}$$

$$\operatorname{grad} \Pi(\mathbf{x}) \in T_{\mathbf{x}}\mathcal{S}^{n-1} \qquad \operatorname{Hess} \Pi(\mathbf{x}) \in (T_{\mathbf{x}}\mathcal{S}^{n-1} \times T_{\mathbf{x}}\mathcal{S}^{n-1})$$

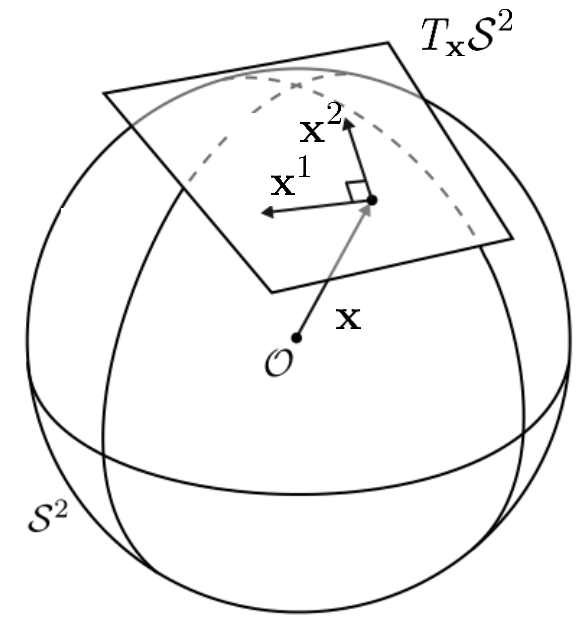
Define a basis of the tangent space in $\mathcal{S}^2$ at $\mathbf{x}$

$$\mathbf{\Lambda} = [\mathbf{x}_I^1 \ \mathbf{x}_I^2] \in \mathbb{R}^{3\times 2}$$

$$\operatorname{grad}^{\text{red}} \Pi(\mathbf{x})_{2\times 1} = \mathbf{\Lambda}^T (P_{\mathbf{x}} \operatorname{grad} \bar{\Pi}(\mathbf{x})) = \mathbf{\Lambda}^T [\operatorname{grad} \bar{\Pi}(\mathbf{x})]_{3\times 1}$$

$$\operatorname{Hess}^{\text{red}} \Pi(\mathbf{x})_{2\times 2} = \mathbf{\Lambda}^T [\operatorname{Hess} \bar{\Pi}(\mathbf{x})]_{3\times 3} \mathbf{\Lambda}$$

Resulting optimization algo has only $n-1$ dofs (as the dim. of the manifold)

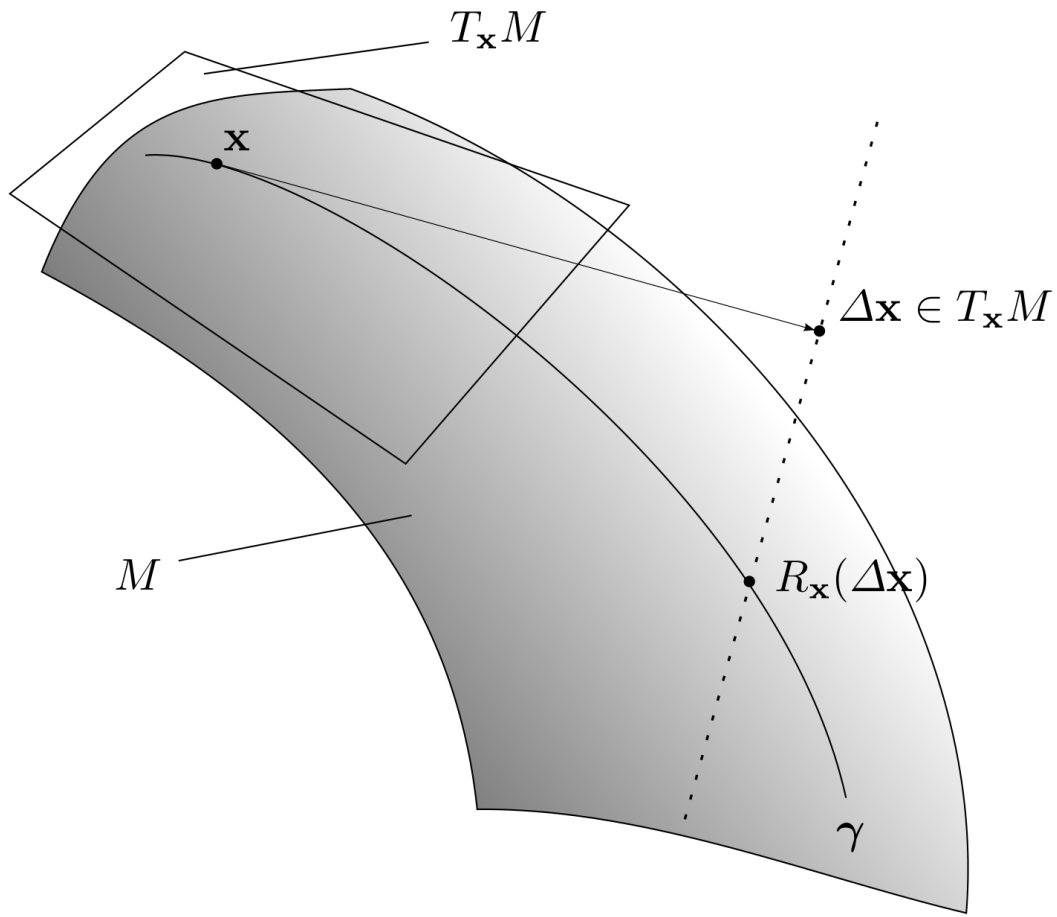


Update of nodal values

Optimization on manifolds

Update of nodal values

$\mathbf{x}_k + \Delta\mathbf{x}_k \notin \mathcal{M}$



The exponential map creates the *unique* geodesic curve starting at $\mathbf{x}_k$ in direction $\Delta\mathbf{x}_k$ with constant speed

$$\gamma(t) = \exp_{\mathbf{x}_k}(t\Delta\mathbf{x}_k) \quad \Longrightarrow \quad \mathbf{x}_{k+1} = \exp_{\mathbf{x}_k}(\Delta\mathbf{x}_k)$$

Along these geodesics one could perform e.g. gradient descent

Algebraic optimization on manifolds

Baustatik und Baudynamik

Update of nodal values

ABSIL PA , "OPTIMIZATION ON MANIFOLDS: METHODS AND APPLICATIONS", LEUVEN, 18 SEP 2009.
LUENBERGER, D.G. (1973) INTRODUCTION TO LINEAR AND NONLINEAR PROGRAMMING. ADDISON-WESLEY, BOSTON.

Luenberger (1973), Introduction to linear and nonlinear programming.
Luenberger mentions the idea of performing line search along geodesics, "*which we would use if it were computationally feasible (which it definitely is not)*".

Generalize the concept of the exponential map → Retractions

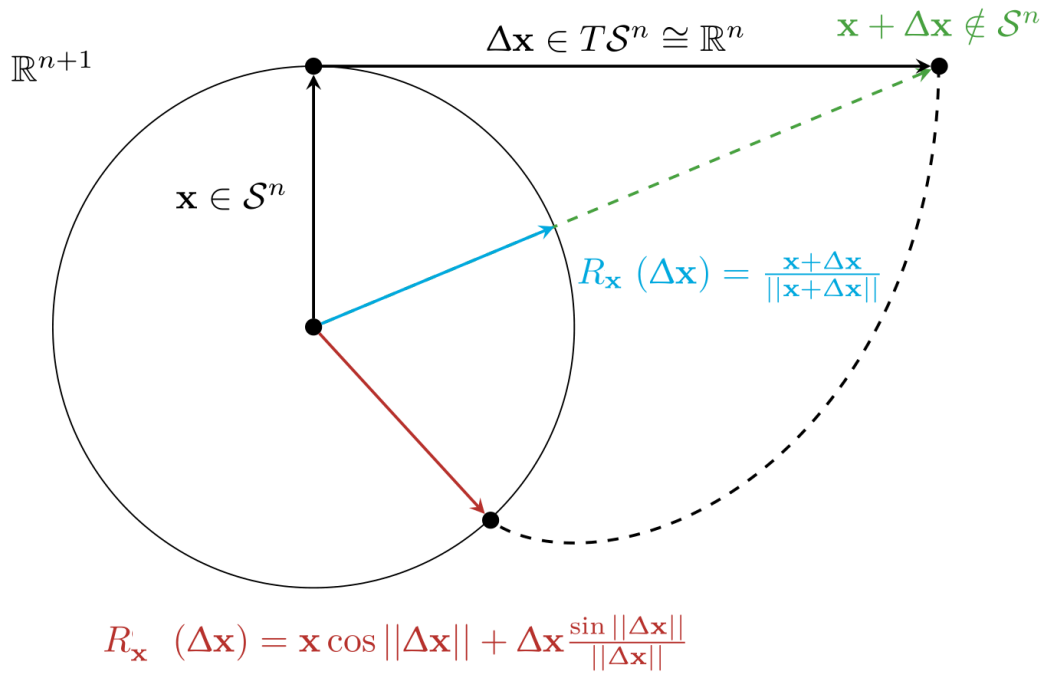
Algebraic optimization on manifolds

Update of nodal values

Retractions for the unit sphere

$$R_{\mathbf{x}}^{\exp}(\Delta\mathbf{x}) = \exp_{\mathbf{x}}(\Delta\mathbf{x}) = \cos(||\Delta\mathbf{x}||)\mathbf{x} + \frac{\sin(||\Delta\mathbf{x}||)}{||\Delta\mathbf{x}||}\Delta\mathbf{x}$$

$$R_{\mathbf{x}}^{\mathrm{rrn}}(\Delta\mathbf{x}) = \frac{\mathbf{x}+\Delta\mathbf{x}}{||\mathbf{x}+\Delta\mathbf{x}||}$$



Update of nodal values

Retractions for the unit sphere

Motivation

Baustatik und Baudynamik

Riemannian Newton

	Classic Newton	Riemannian Newton
Update	$\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta\mathbf{x}_k$	$\mathbf{x}_{k+1} = R_{\mathbf{x}_k}(\Delta\mathbf{x}_k)$
Gradient	$\operatorname{grad} f(\mathbf{x})$	$\operatorname{grad} f(\mathbf{x}) = P_{\mathbf{x}} \operatorname{grad} \bar{f}(\mathbf{x})$
Hessian	$\operatorname{Hess} f(\mathbf{x})$	$\operatorname{Hess} f(\mathbf{x})\boldsymbol{\eta} = P_{\mathbf{x}} \operatorname{Hess} \bar{f}(\mathbf{x}) P_{\mathbf{x}}\boldsymbol{\eta} + W_{\mathbf{x}}(\boldsymbol{\eta}, P_{\mathbf{x}}^{\perp} \operatorname{grad} \bar{f}(\mathbf{x}))$

Ingredients:

$R_{\mathbf{x}} : T_{\mathbf{x}}\mathcal{M} \to \mathcal{M}$

$\mathbf{P}_{\mathbf{x}} : \mathbb{R}^n \to T_{\mathbf{x}}\mathcal{M}$

$W_{\mathbf{x}} : T_{\mathbf{x}}\mathcal{M} \times T_{\mathbf{x}}^{\perp}\mathcal{M} \to T_{\mathbf{x}}\mathcal{M}$

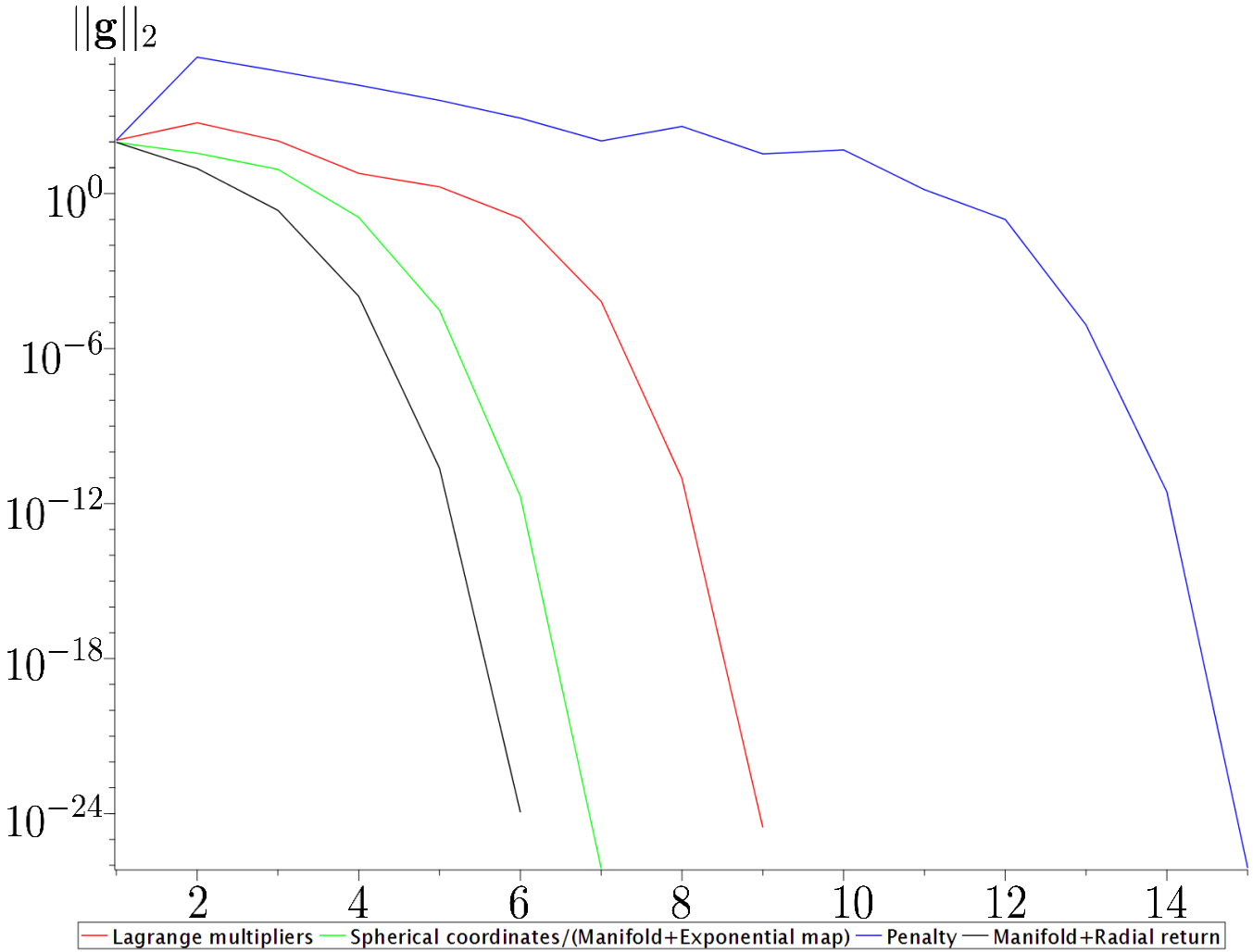
Tangent space basis $\boldsymbol{\Lambda}$

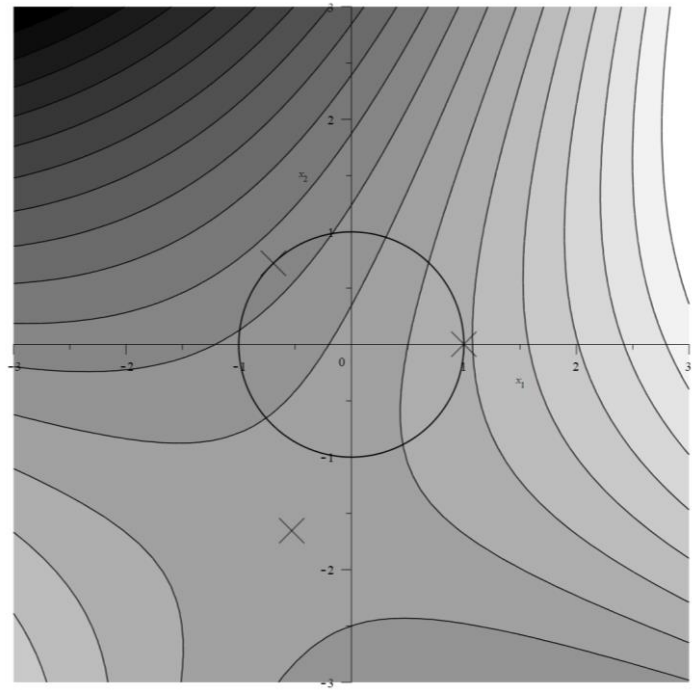
$\operatorname{Hess} \bar{E}(\mathbf{x})$

$\operatorname{grad} \bar{E}(\mathbf{x})$

Algebraic optimization on manifolds

Toy problem: Newton's method, iteration count vs. gradient norm





#iteration

Optimization on manifolds

Baustatik und Baudynamik

Summary

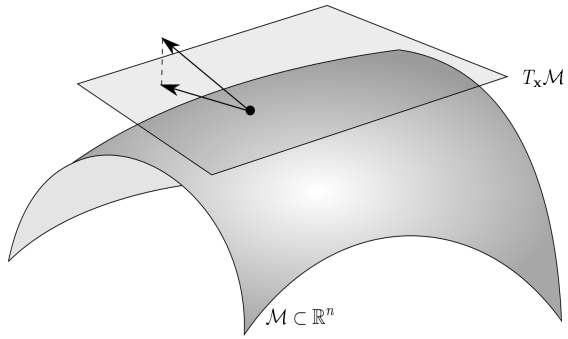
	LAM	Penalty	Coords	Manifold optimization
Linearization	☺	☺	☹	☺
Singularities	☺	☺	☹	☺
Search space dimensions	3	2	1	1
Minimization	☹	☺	☺	☺
Iterations	☹	☹	☺	☺

Outline

Baustatik und Baudynamik

Interpolation on manifolds
(FE for manifolds)

Optimization on manifolds



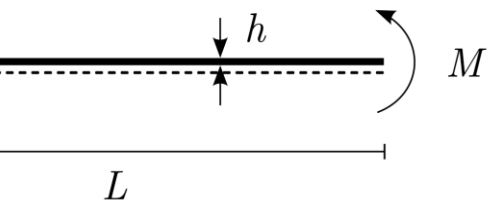
Simulations

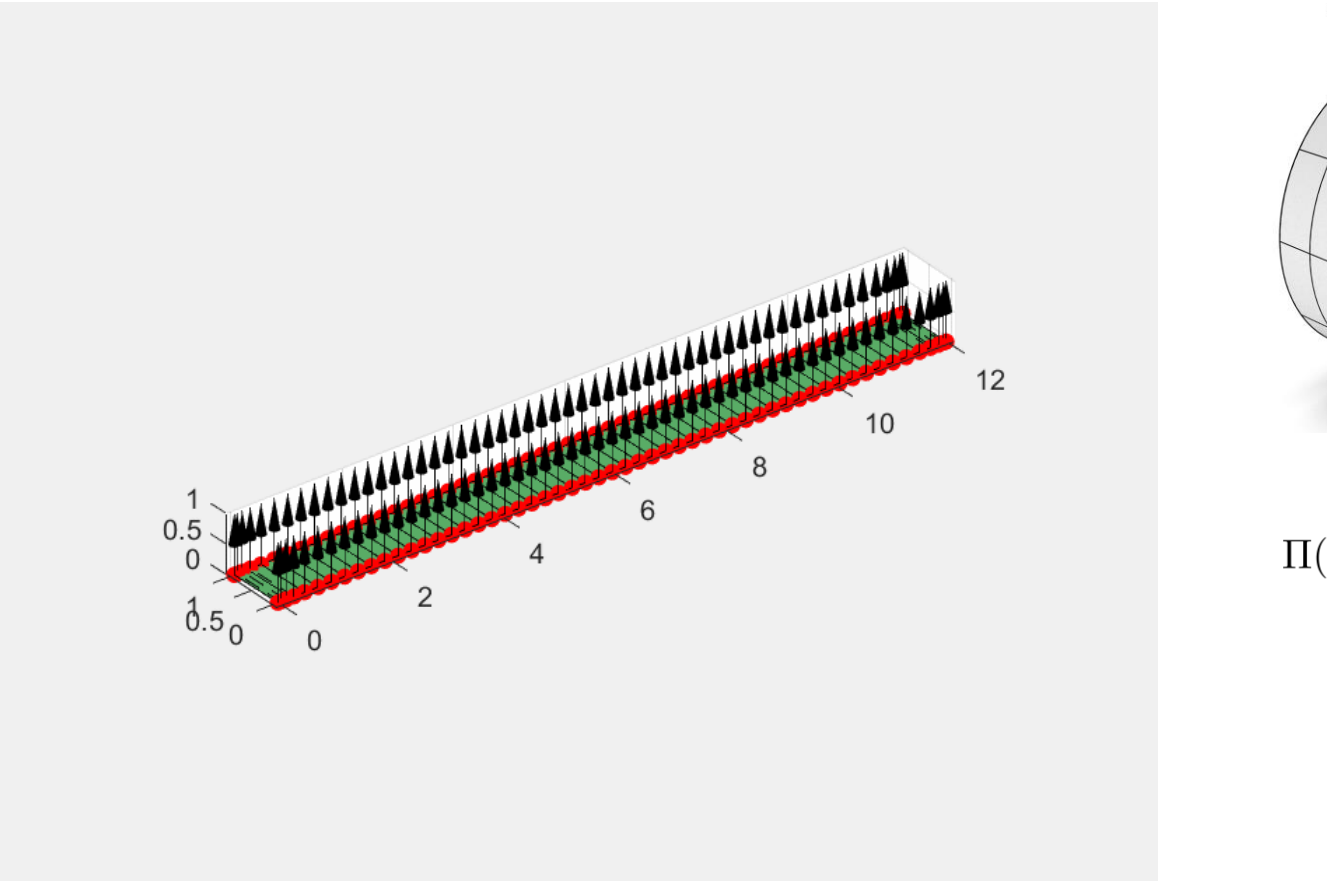
Open questions/Outlook

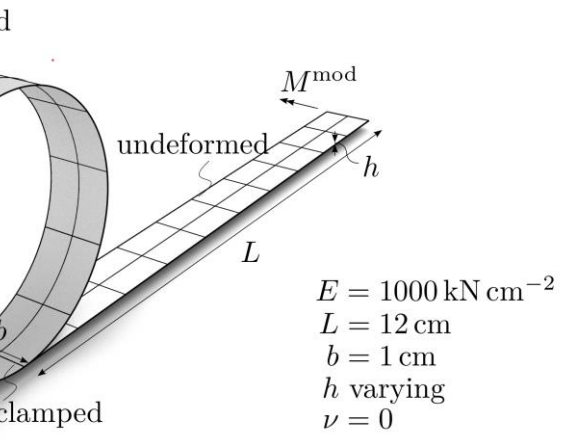
Simulations

Reissner-Mindlin: Roll-up of clamped beam

- 16 Iterations of Newton's method to reach equilibrium





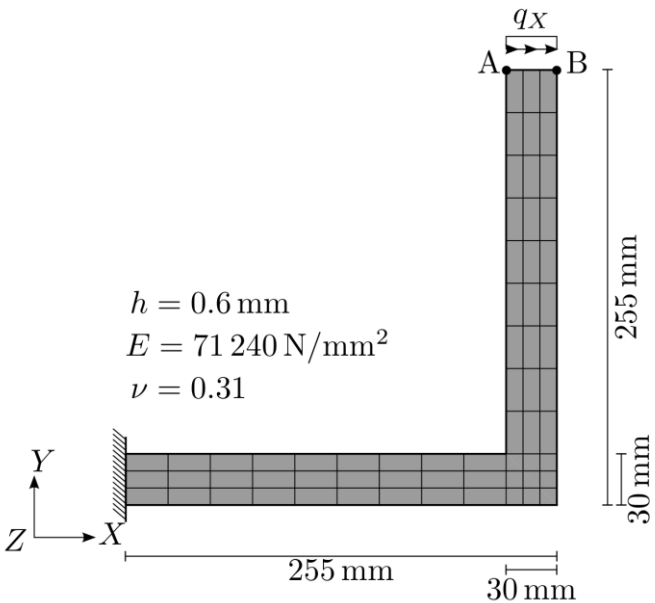


$$\Pi(\boldsymbol{\varphi}, \mathbf{t}) = \int_{\mathcal{B}} \psi_{\mathrm{strain}}(\boldsymbol{\varphi}, \mathbf{t})\,\mathrm{d}\,V + \Pi_{\mathrm{ext}}(\boldsymbol{\varphi}, \mathbf{t})$$

$$\{\boldsymbol{\varphi}^*, \mathbf{t}^*\} = \arg\left\{\min_{\boldsymbol{\varphi}\in\mathbb{R}^d}\ \min_{\mathbf{t}\in\mathcal{S}^{d-1}} \Pi(\boldsymbol{\varphi}, \mathbf{t})\right\}.$$

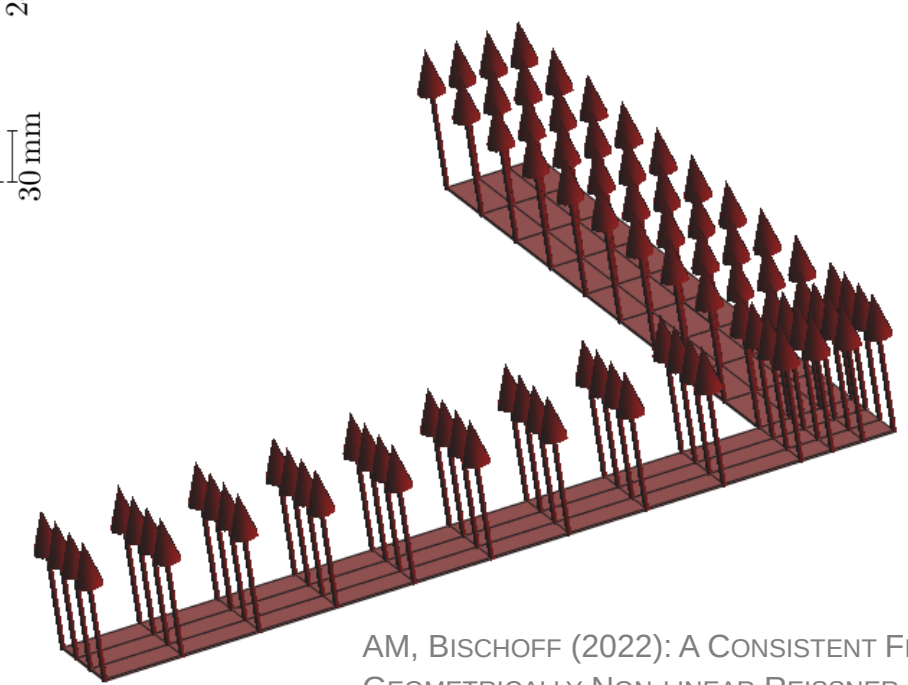
Simulations

Reissner-Mindlin: L-shape



$$\Pi(\boldsymbol{\varphi}, \mathbf{t}) = \int_{\mathcal{B}} \psi_{\mathrm{strain}}(\boldsymbol{\varphi}, \mathbf{t})\,\mathrm{d}\,V + \Pi_{\mathrm{ext}}(\boldsymbol{\varphi}, \mathbf{t})$$

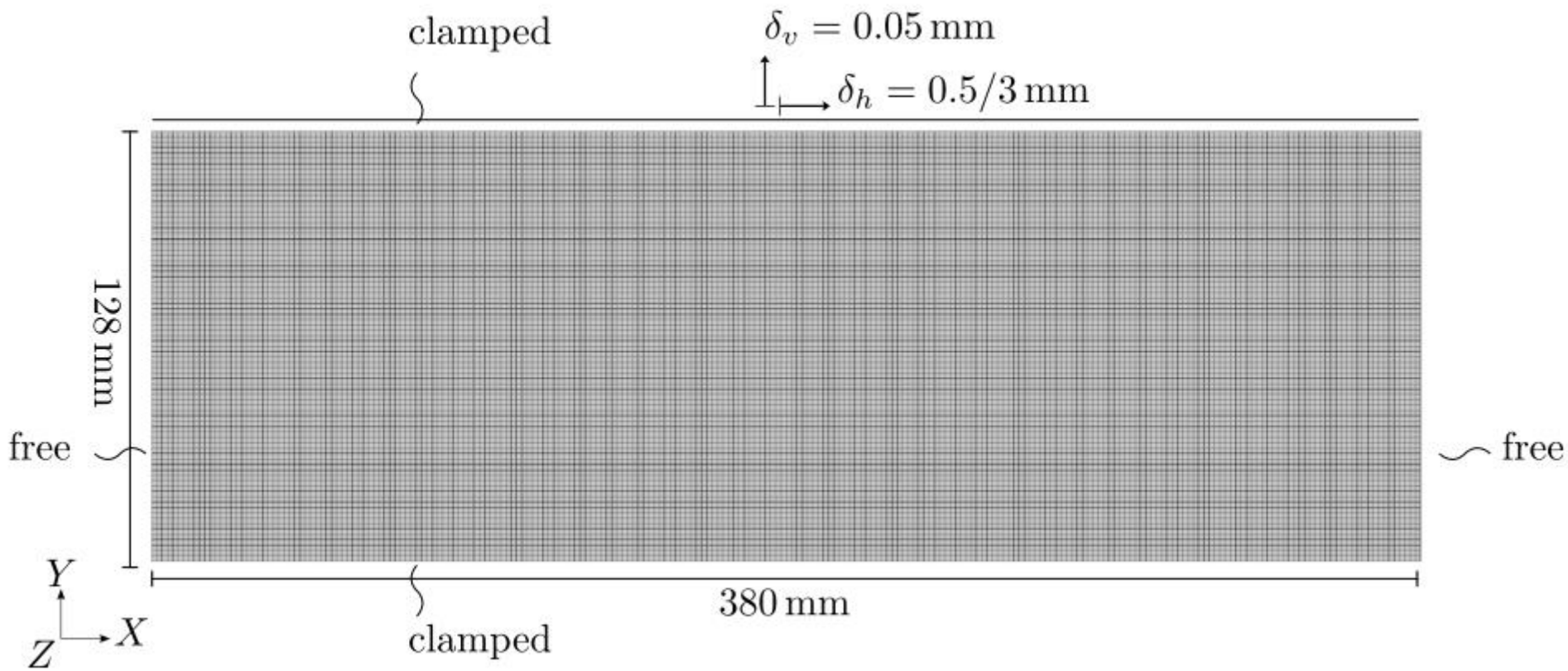
$$\{\boldsymbol{\varphi}^*, \mathbf{t}^*\} = \arg\left\{\min_{\boldsymbol{\varphi}\in\mathbb{R}^d}\ \min_{\mathbf{t}\in\mathcal{S}^{d-1}} \Pi(\boldsymbol{\varphi}, \mathbf{t})\right\}.$$



AM, Bischoff (2022): A Consistent Finite Element Formulation of the Geometrically Non-linear Reissner-Mindlin Shell Model, DOI

Simulations

Reissner-Mindlin: Buckling of a sheared sheet



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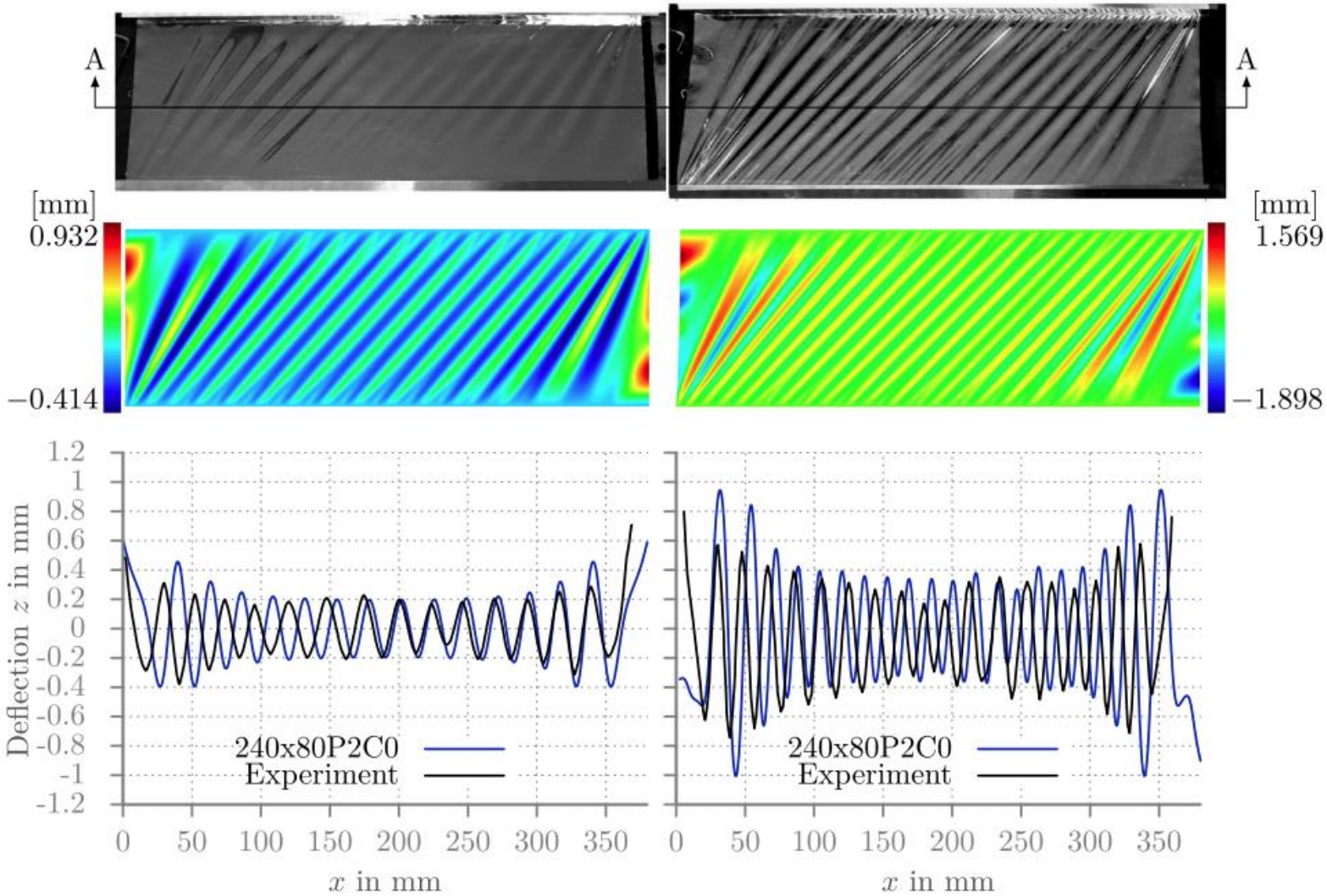
Simulations

Reissner-Mindlin: Buckling of a sheared sheet



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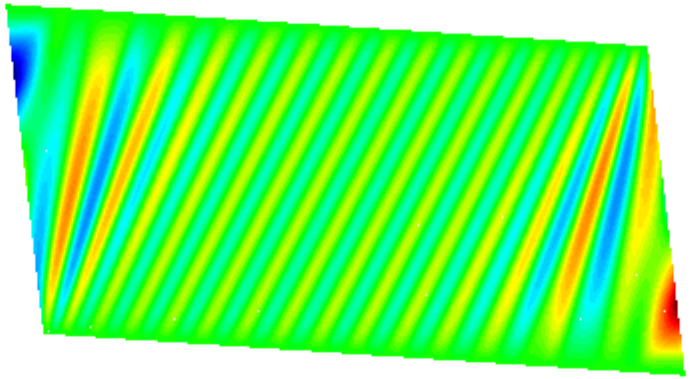
Wong W, Pellegrino S (2006) Wrinkled membranes Part I: Experiments. J Mech Mater Struct 1:3–25. DOI

Simulations

Reissner-Mindlin: Buckling of a sheared sheet

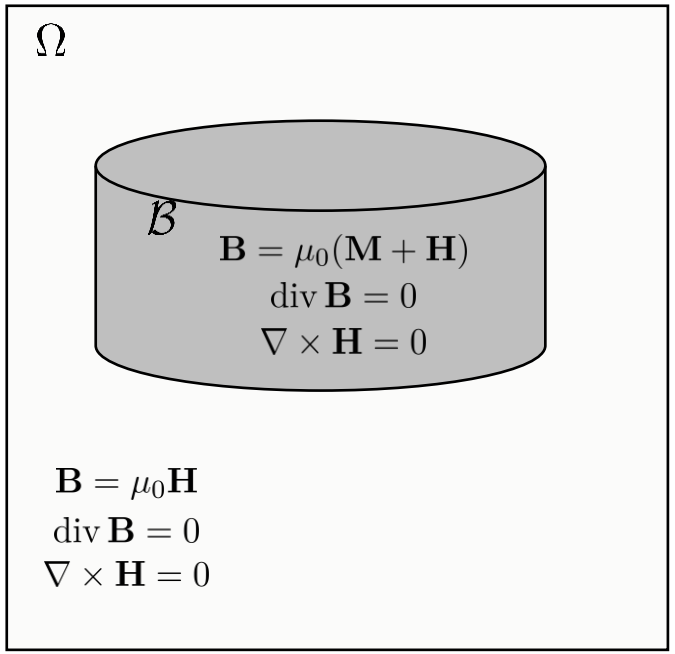


AM, Bischoff (2022): A Consistent Finite Element Formulation of the Geometrically Non-linear Reissner-Mindlin Shell Model, DOI

Other physical problems

Simulation of micromagnetics

$$\Pi(\mathbf{a},\mathbf{m}) = \int\limits_{\Omega\setminus\mathcal{B}} \frac{1}{2}||\operatorname{curl}\mathbf{a}||^2 \,\mathrm{d}\,V + \int\limits_{\mathcal{B}} \frac{1}{2}||\operatorname{grad}\mathbf{m}||^2 + \frac{1}{2}||\operatorname{curl}\mathbf{a}||^2 - \operatorname{curl}\mathbf{a}\cdot\mathbf{m}\,\mathrm{d}\,V$$



$$\mathbf{m} \in \mathcal{S}^2$$

$$\mathbf{a} \in \mathbb{R}^3$$

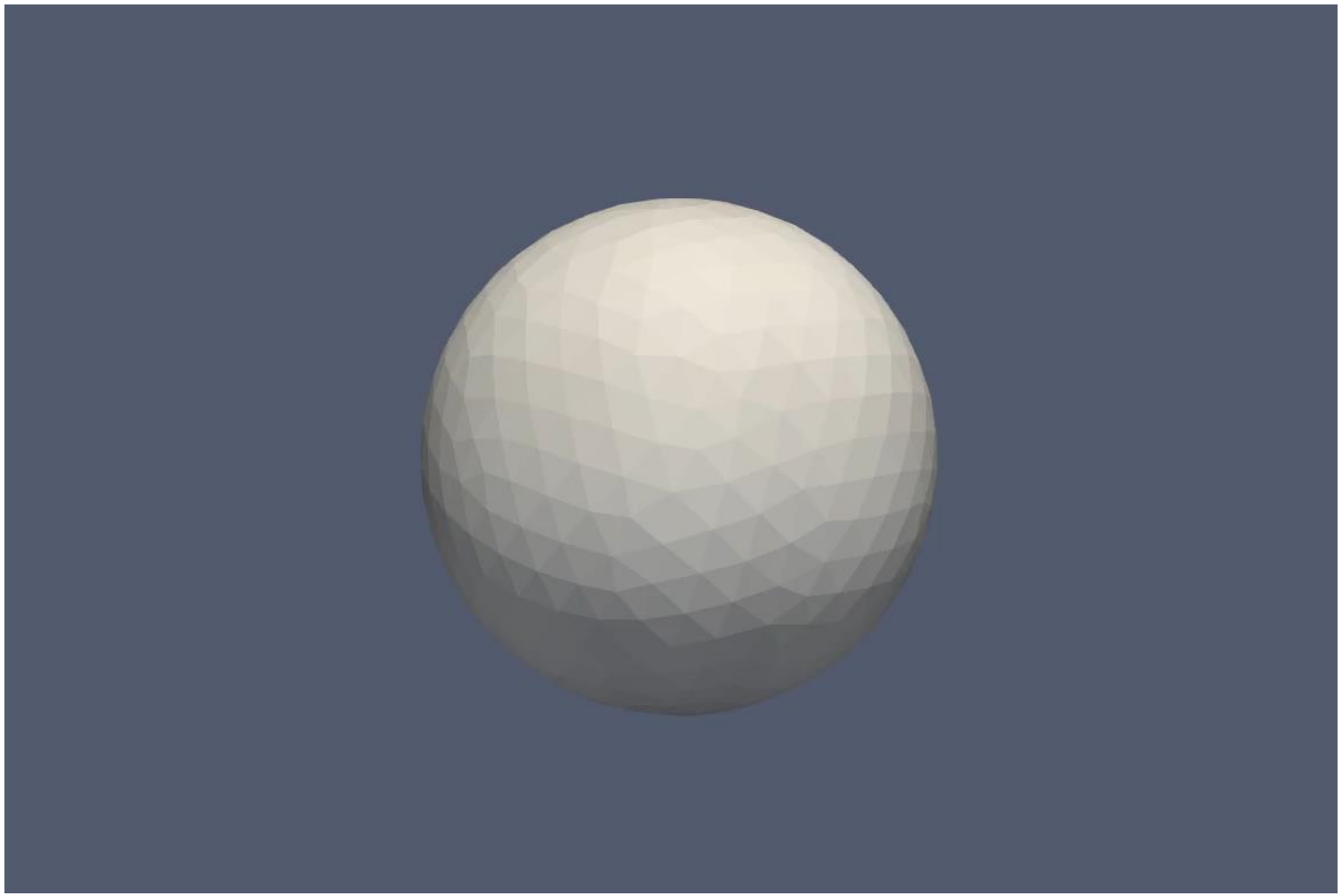
$$\mathbf{b} = \nabla\times\mathbf{a}$$

$$\min_{\mathbf{m}\in\mathcal{S}^2}\ \min_{\mathbf{a}\in\mathbb{R}^3} \Pi(\mathbf{m},\mathbf{a})$$

Maxwell's equation in vacuum and matter

Other physical problems

Simulation of micromagnetics



Open questions

Other manifolds

Manifold	Exponential map	Other retractions	Tangent space
Unit sphere $\mathcal{S}^{n-1}$	$\cos\|\|\Delta\mathbf{x}\|\|\mathbf{x} + \frac{\sin\|\|\Delta\mathbf{x}\|\|}{\|\|\Delta\mathbf{x}\|\|}\Delta\mathbf{x}$	$\frac{\mathbf{x}+\Delta\mathbf{x}}{\|\|\mathbf{x}+\Delta\mathbf{x}\|\|}$	$T_{\mathbf{x}}\mathcal{S}^{n-1} = \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y}^T\mathbf{x} = 0\}$
Special linear group $\mathcal{SL}(n) = \{\mathbf{X} \in \mathbb{R}^{n\times n} \,\|\, \det \mathbf{X} = 1\}$	$\exp(\Delta\mathbf{X})\mathbf{X}$	$\frac{\mathbf{X}+\Delta\mathbf{X}}{\det(\mathbf{X}+\Delta\mathbf{X})^{1/n}}$	$T_{\mathbf{X}}\mathcal{SL}(n) = \{\mathbf{Y} \in \mathbb{R}^{n\times n} \,\|\, \operatorname{tr}\mathbf{Y} = 0\}$
Symmetric Special linear manifold $\mathcal{SSL}(n) = \{\mathbf{X} \in \mathbb{R}^{n\times n} \,\|\, \det\mathbf{X} = 1 \wedge \mathbf{X}^T = \mathbf{X}\}$	$\exp(\Delta\mathbf{X})\mathbf{X}$	$\frac{\mathbf{X}+\Delta\mathbf{X}}{\det(\mathbf{X}+\Delta\mathbf{X})^{1/n}}$	$T_{\mathbf{X}}\mathcal{SSL}(n) = \{\mathbf{Y} \in \mathbb{R}^{n\times n} \,\|\, \operatorname{tr}(\mathbf{X}^{-1}\mathbf{Y}) = 0\}$
Special orthogonal group $\mathcal{SO}(n) = \{\mathbf{X} \in \mathbb{R}^{n\times n} \,\|\, \mathbf{X}^T\mathbf{X} = \mathbf{I} \wedge \det\mathbf{X} = 1\}$	$\exp(\Delta\mathbf{X})\mathbf{X}$	QR decomposition	$T_{\mathbf{X}}\mathcal{SO}(n) = \{\mathbf{Y} \in \mathbb{R}^{n\times n} \,\|\, \mathbf{Y}^T = -\mathbf{Y}\}$

Outlook

Baustatik und Baudynamik

Apply results

- Geometrically nonlinear beam

Dynamics on manifolds

- Non-constant mass matrix
- Riemannian Hamiltonian/Lagrangian
- Variational integrators for manifolds
- …

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Boumal N (2020) *An introduction to optimization on smooth manifolds*. Available online, URL Link to online resource

Finite elements for manifolds:

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Sander O (2012) *Geodesic finite elements on simplicial grids*. Int J Numer Methods Eng 92(12):999–1025, doi:10.1002/nme.4366

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Grohs P, Hardering H, Sander O, Sprecher M (2019) Projection-based fnite elements for nonlinear function spaces. SIAM J Numer Anal, doi: 10.1137/18M1176798

Hardering H (2018) *L2-discretization error bounds for maps into Riemannian manifolds*. Numer Math (Heidelb) 139(2):381–410, doi:10.1007/s00211-017-0941-3

Physical simulations (Small sample):

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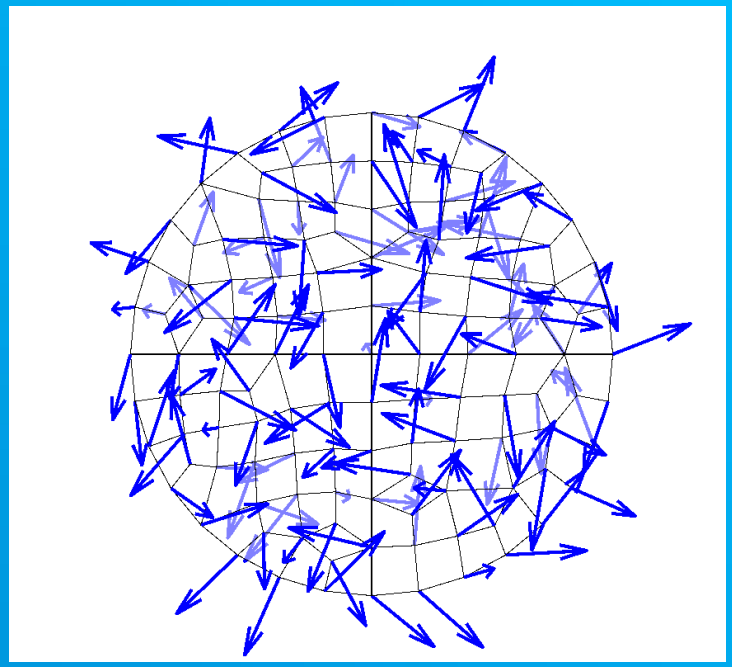
Universität Stuttgart



Thank you!

Alex Müller

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Slides https://www.ibb.unistuttgart.de/institut/team/Mueller-00006/

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